D'Alembert's Principle

Consider a system of n particles and K equation of holonomic constraints.

Static case: Assume a system is in equilibrium $\vec{F}_i = 0 \quad (i = 1 \cdots n)$

$\vec{F}_i = \vec{F}_i^a + \vec{f}_i$ = applied forces + forces of constraints

Consider a virtual displacement of the system, i.e. an infinitesimal change in the coordinates of the system, denoted by $\delta \vec{r}_i$, consistent with the constraints imposed on the system at a given instant t.

The work done by the forces in the virtual displacement $\delta \vec{r}_i$ is called the virtual work.

For a system in equilibrium the virtual work is zero.

$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$ Since the virtual displacements are compatible with the forces of constraints, $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$, and therefore $\sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = 0$ The virtual work of the applied force vanishes for a system in equilibrium.

Dynamic case: If the system is moving we have $\vec{F}_i - \dot{\vec{p}}_i = 0$

$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$. As the total work of the forces of constraints vanishes we have D'Alembert's Principle $\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$.

This equation is the starting point for obtaining equations of motion for the generalized coordinates, since all constraints have disappeared from this equation.

Lagrange's equations

Assume a system of n particles and k equations of holonomic constraints. Introduce s generalized coordinates $q_1, q_2, \cdots, q_s$, $s = 3n - k$. The generalized coordinates are independent.

We have

$$\vec{r}_i = \vec{r}_i(q_1, \cdots, q_s, t).$$

Therefore

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_{k=1}^{s} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad \frac{\partial \vec{v}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

$$\delta \vec{r}_i = \sum_{k=1}^{s} \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$$

since $\delta \vec{r}_i$ is an infinitesimal displacement made at a fixed time.

D'Alembert's principle can be rewritten.

$$\sum_{i=1}^{n} \vec{F}_i^{a} \cdot \delta \vec{r}_i = \sum_{k=1}^{s} Q_k \delta q_k$$

$$Q_k = \sum_{i=1}^{n} \vec{F}_i^{a} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

The quantities Q_k are called the generalized applied forces (They need not have the dimension of force)

$$\sum_{i=1}^{n} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{n} m_i \sum_{k=1}^{s} \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$$

rewrite $\ddot{\vec{r}}_i \cdot \dfrac{\partial \vec{r}_i}{\partial q_k} = \dfrac{d}{dt}\left(\dot{\vec{r}}_i \cdot \dfrac{\partial \vec{r}_i}{\partial q_k}\right) - \dot{\vec{r}}_i \cdot \dfrac{d}{dt} \dfrac{\partial \vec{r}_i}{\partial q_k}$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \sum_{j=1}^{s} \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j} \dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial q_k \partial t} = \frac{\partial \vec{v}_i}{\partial q_k}$$

Therefore $\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_K} = \sum_{i=1}^{n} \left[m_i \frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_K}\right) - m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_K} \right]$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_K}\left(\sum_{i=1}^{n} \frac{m_i v_i^2}{2}\right)\right] - \frac{\partial}{\partial q_K}\left(\sum_{i=1}^{n} \frac{m_i v_i^2}{2}\right)$$

Finally

$$\sum_{i=1}^{n} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{K=1}^{s} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K}\right] \delta q_K$$

and D'Alemberts principle becomes:

$$\sum_{K=1}^{s} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} - Q_K\right] \delta q_K = 0$$

Since the generalized coordinates are independent coordinates the δq_K are independent and each term must vanish individually. Thus we obtain the set of equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = Q_K} \quad , \quad K = 1 \cdots s \; .$$

If there exists some scalar function $U(q)$, so that all applied force are given by $Q_K = -\frac{\partial U}{\partial q_K}$, or some scalar function $U(q, \dot{q})$, so that all the applied forces are given by $Q_K = -\frac{\partial U}{\partial q_K} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_K}$, then the equations of motion

can be obtained from Lagrange's equations.

$$L = T - U \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1 \cdots s$$

L is called the Lagrangian of the system.

Note: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = Q_K$ holds as long as we have holonomic constraints, and therefore independent coordinates.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) - \frac{\partial L}{\partial q_K} = 0$ holds, if in addition, we have potential forces.

For conservative forces we have $\vec{F}_i = -\vec{\nabla}_i U(\vec{r}_1, \cdots \vec{r}_n)$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla} V_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$\frac{\partial U(\vec{r}_1, \cdots \vec{r}_n)}{\partial q_j} = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial q_j} + \frac{\partial U}{\partial y_1}\frac{\partial y_1}{\partial q_j} + \frac{\partial U}{\partial z_1}\frac{\partial z_1}{\partial q_j} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial q_j} + \frac{\partial U}{\partial y_2}\frac{\partial y_2}{\partial q_j}$$

$$+ \frac{\partial U}{\partial z_2}\frac{\partial z_2}{\partial q_j} + \cdots\cdots + \frac{\partial U}{\partial x_n}\frac{\partial x_n}{\partial q_j} + \frac{\partial U}{\partial y_n}\frac{\partial y_n}{\partial q_j} + \frac{\partial U}{\partial z_n}\frac{\partial z_n}{\partial q_j}$$

$$= \sum_i \vec{\nabla}_i U \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

therefore $Q_j = -\frac{\partial U}{\partial q_j}$

As long as we have independent coordinates and conservative forces Lagrange's equations hold.