August 2023 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^8$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider two particles with mass m. The first particle, with energy E, collides elastically with the second particle, which is at rest.

(a) If both scatter at an angle θ with respect to the incoming particle's momentum, write down the value of θ in terms of E and m.

(b) What value does θ approach in the relativistic limit?

(c) What value does θ approach in the non-relativistic limit?

Problem 2:

Consider a spin coupled to a magnetic field via the Hamiltonian $H = \gamma\mathbf{S}\cdot\mathbf{B}$, where the three-dimensional magnetic field is $\mathbf{B} = (B_x, B_y, B_z) = B_0(1, 0, 0)$, B_0 is a constant, and γ is another constant. The three-dimensional spin vector is $\mathbf{S} = (S_x, S_y, S_z)$, and the S_x, S_y, S_z are the usual 2 x 2 spin matrices.

(a) In the eigenbasis of S_z, find the matrix of H and its eigenvalues.

(b) In the eigenbasis of S_z, find the two normalized eigenvectors of the H.

Problem 3:

Consider a system of N distinguishable non-interacting spins in a magnetic field **B**. Each spin has a magnetic moment of magnitude μ, and each can point either parallel or antiparallel to the field. Thus, the energy of a particular state is $\sum_{n=1}^{N}(-n_i\mu B)$, $n_i = \pm 1$.

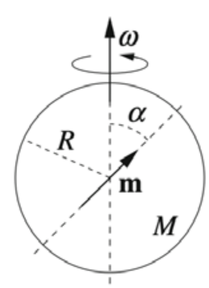
(a) Determine the thermodynamically defined internal energy U of this system as a function of $\beta = 1/kT$, B, and N by employing an ensemble characterized by these variables.

(b) Determine the entropy S of this system as a function of β, B, and N.

Stirling approximation for N >> 1: $\ln(N!) \approx N\ln(N) - N$.

Problem 4:

A pulsar is a neutron star with mass $M \approx 1.4 * M_{sun} \approx 2.8 \times 10^{30}$ kg and radius $R \approx 10$ km. The star rotates with angular velocity ω and has a magnetic moment **m**, which is, in general, not parallel to the rotation axis.



(a) Describe the radiation emitted by the pulsar, and find the total radiated power, assuming that the angle between the magnetic moment and the rotation axis is α, as in the figure.

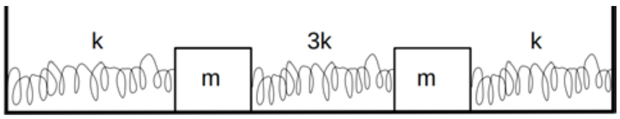
(b) Find the "spin-down rate" (i.e. find $\omega(t)$) of the pulsar, assuming that energy loss is due to radiation only.

Problem 5:

The Lagrangian for the system below can be written as

$$L = \frac{1}{2}\begin{bmatrix}\dot{x}_1 & \dot{x}_2\end{bmatrix}[T]\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}x_1 & x_2\end{bmatrix}[K]\begin{bmatrix}x_1 \\ x_2\end{bmatrix}.,$$

where x_i denotes the displacement of mass m_i from its equilibrium position.



(a) Find the matrices T and K.

(b) Find the eigenvalues and eigenvectors for the oscillations of this system.

(c) Write down the general solution for $x_1(t)$ and $x_2(t)$.

(d) For the initial conditions $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$, $\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}1 \\ -1\end{bmatrix}$, solve for the constants in your answer for (c) and determine for $x_1(t)$ and $x_2(t)$ for these initial values.

Problem 6:

A planetesimal orbiting the Sun absorbs solar energy at a rate equal to the solar flux at the orbital radius multiplied by the cross-sectional area of the planetesimal multiplied by (1 - the albedo). The temperature of the planetesimal can be estimated by setting the rate of solar energy absorption equal to the rate at which the planetesimal is radiating energy.

(a) If we assume the planetesimal radiates as a black body, and the solar luminosity is 3.8×10^{26} Watts, calculate the temperatures of planetesimals as a function of distance from the Sun and albedo.

(b) For water ice, with an albedo of 0.35 and sublimation temperature of 200 K, find the minimum distance from the Sun that a water ice planetesimal can remain solid.

For reference, the Earth orbital radius is $1.5*10^{11}$ m.

Problem 7:

A particle of mass m is constrained to move in an infinitely deep, one-dimensional square well extending from 0 to +a.

If this particle is under the influence of a delta-function perturbation $H' = -\alpha\delta(x - a/2)$, where α is a constant.

(a) (i) Find the first-order correction to the allowed energies.

(ii) Explain why the energies are not perturbed for even n.

(b) Find the first three nonzero correction terms in the first-order expansion of the ground state wave function.

Problem 8:

A particle of mass m moves under the influence of gravity on the inner surface of a paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless.

(a) Introduce cylindrical coordinates ρ, ϕ, z. Write a Lagrangian for the system employing ρ and ϕ as generalized coordinates.

(b) Find two constants of motion.

(c) Obtain the equations of motion.

(d) Show that the particle will describe a horizontal circle in the plane $z = h$ provided that it is given the proper angular velocity ω. What is the magnitude of this velocity?