August 2024 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\, 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Coordinate system S' is rotating with a time-dependent angular velocity $\boldsymbol{\omega}$ relative to a fixed coordinate system S. The relationship between the acceleration **a** of a point in S and the acceleration **a**' of a point in S' is given by

$$\mathbf{a} = \mathbf{a}' + d\boldsymbol{\omega}/dt \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

(a) Indicate what fictious forces each of the last three terms correspond to.

(b) A particle is projected vertically upward with velocity v_0 on Earth's surface at a northern latitude λ. Where will the particle land with respect to its launch location?
For this problem, assume the launch height is small compared to Earth's radius R.
State any other assumptions or approximations you make explicitly.

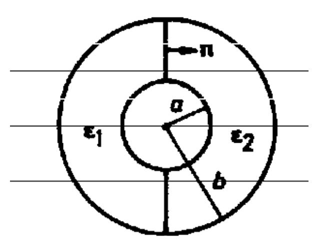
Problem 2:

For a one-dimensional simple harmonic oscillator, consider a correlation function defined as $C(t) = \langle \hat{x}(t)\hat{x}(0)\rangle$, where $\hat{x}(t) = \hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t$, with $\hat{x}, \hat{p}$ being the position and momentum operators.
Calculate $C(t)$ explicitly for the ground state of a one-dimensional simple harmonic oscillator, i.e., $C(t) = \langle 0|\hat{x}(t)\hat{x}(0)|0\rangle$, where $|n\rangle$ refers to the eigenstate of a simple harmonic oscillator with energy $\left(n+\frac{1}{2}\right)\hbar\omega$.

Hint: $\hat{x}$ and $\hat{p}$ can be expressed as $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)$, $\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}-\hat{a}^+)$, where $\hat{a}$ and $\hat{a}^+$ are the annihilation and creation operators that behave as $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$.

Problem 3:

One half of the region between the plates of a spherical capacitor of inner and outer radii a and b is filled with a linear isotropic dielectric of permittivity ε_1 and the other half has permittivity ε_2, as shown in figure. If the inner plate has total charge Q and the outer plate has total charge -Q, find



(a) the electric displacements $\mathbf{D}_1$ and $\mathbf{D}_2$ in the region of ε_1 and ε_2,

(b) the electric fields in the region of ε_1 and ε_2,

(c) and the total capacitance of this system.

Problem 4:

Consider the matrices $\hat{A} = \begin{pmatrix} 1 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 1 \end{pmatrix}$ and $\hat{B} = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 1 \end{pmatrix}$.

(a) Argue why both matrices are diagonalizable. No calculation allowed.

(b) Find the eigenvalues and eigenvectors of $\hat{A}$.

(c) Find $[\hat{A}, \hat{B}]$.

(d) The similarity transformation $\hat{B}' = \hat{Q}^{-1}\hat{B}\hat{Q}$ diagonalizes matrix $\hat{B}$. Find $\hat{Q}$ and $\hat{Q}^{-1}$.

Problem 5:

A zipline can be modelled by a block of mass M on a frictionless inclined plane (at an angle α with respect to flat). This block then has a pendulum (with a massless rigid rod of length L and a bob of mass m) attached to it hanging below.

(a) Find the equations of motion and any equilibrium behaviors.

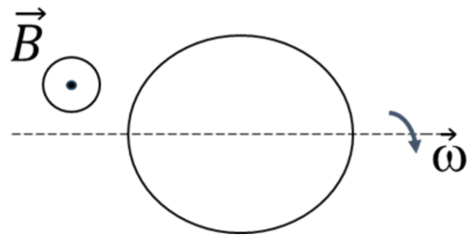
(b) Then, assuming small oscillations about any stable equilibria, find the frequencies of any normal modes.

Problem 6:

An object of rotational inertia I is initially at rest. A torque is then applied causing the object to begin rotating. The torque is applied for only one-quarter of a revolution, during which time its magnitude is given by $\tau = A\cos\theta$, where A is a constant and θ is the angle through which the object has rotated. What is the final angular speed of the object?

Problem 7:

A conducting loop of radius a, resistance R, and moment of inertia I is rotating around an axis in the plane of the loop, initially at an angular frequency ω_0. A uniform static magnetic field **B** is applied perpendicular to the rotation axis.



(a) What is the initial kinetic energy of the loop?

(b) Calculate the rate of kinetic energy dissipation, assuming it all goes into Joule heating of the loop resistance.

(c) In the limit that the change in energy per cycle is small, derive the differential equation that describes the time dependence of the angular velocity . How long will it take for ω to fall to 1/e of its initial value?

Hint: In the above limit you can replace the instantaneous rate of energy dissipation by its average value over a cycle.

Problem 8:

(a) One mole of ideal gas with constant heat capacity C_V is placed inside a cylinder. Inside the cylinder there is a piston which can move without friction along the vertical axis. Pressure P_1 is applied to the piston and the gas temperature is T_1.

At some point, P_1 is **abruptly** changed to P_2 (e.g. by adding or removing a weight from the piston). As a result, the gas volume changes adiabatically. Find the temperature T_2 and the volume V_2 after the thermodynamic equilibrium has been reached in terms of C_V, P_1, T_1, and P_2. Use the relation between heat capacities C_V and C_P to simplify the formulas.

Definition of C_V: $dU = C_V dT$, $C_P = C_V + R$

(b) After the thermodynamic equilibrium has been established in part (a), the pressure is abruptly reset to its original value P_1. Compute final values of the temperature T_f and the volume V_f after the thermodynamic equilibrium has been reached again.

Compute the difference in temperatures (T_f - T_1) and show that it is quadratic in (P_2 - P_1).

Comment on the sign of the temperature difference.