January 2025 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Planck radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider a particle with mass m and energy E traveling in one-dimension from $x = -\infty$. Calculate the probability for it to tunnel, i.e. the transmittance T, through a square potential barrier of height $U_0 > E$ located at $0 \le x \le L$.

Problem 2:

A static charge distribution produces a radial electric field

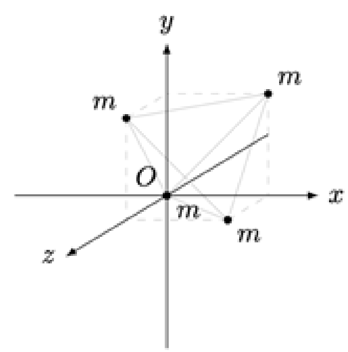
$\mathbf{E} = (A/r^2)\exp(-br)\,\mathbf{e}_r$.

where A and b are constants.

(a) What is the charge density? Sketch it.

(b) What is the total charge Q?

Problem 3:



You have a system of 4 particles of equal mass M = 1 kg. One sits at the origin (x, y, z) = (0, 0, 0) m.

The others sit at (1, 1, 0) m, (0, 1, 1) m, (1, 0, 1) m. These particles are attached to each other by massless rigid rods such that they will always maintain their positions relative to one another.

(a) Calculate the full moment of inertia tensor for this system.

(b) Find the principal axes for this system.

(c) Calculate the angular momentum **L** of this system if you spin it with angular speed ω around the principle axis that corresponds to the smallest eigenvalue.

In case it's useful to you, here's a fact:

$(4 - x)^3 - 3(4 - x) - 2 = (2 - x)(5 - x)^2$.

Problem 4:

In the Aharanov Bohm experiment, a beam of electrons is split in two and they pass on opposite sides of a very long solenoid of radius $r_a < r$. The solenoid is centered at the origin of the x-y plane, is pointing along the z-axis, and is carrying a steady current I. The solenoid is infinitely long and the magnetic field inside is uniform. The magnetic field outside the solenoid is zero. The two beams (wavefunctions) acquire opposite phase shifts as they pass the solenoid along opposite sides on their way to the detector, resulting in an interference pattern. In this problem, we will calculate these phase shifts.

(a) Adopting the Coulomb gauge, show that the vector potential $\boldsymbol{A}$ for $r > r_a$ is given by

$\boldsymbol{A} = \frac{\Phi}{2\pi r}\boldsymbol{e_\phi}$,

where ϕ is the azimuthal angle in the $x - y$ plane and Φ is the magnetic flux through the solenoid.

(b) The classical Hamiltonian for a particle of charge q and momentum $\boldsymbol{p}$ in the presence of an electromagnetic field is given by

$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + V$,

where $\boldsymbol{A}$ is the vector potential and V the scalar potential.

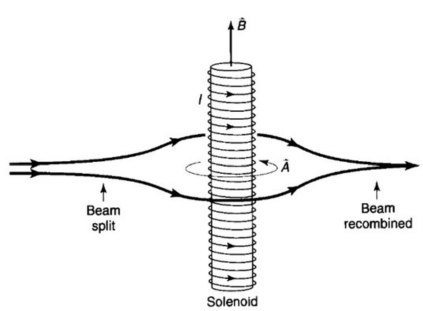
Write down the time-dependent Schrödinger equation for a quantum particle of charge q and wavefunction Ψ in the presence of an electro-magnetic field.

(c) Let's define a new wavefunction $\Psi' \equiv \Psi\, e^{-ig(r)}$ with $g(\boldsymbol{r}) \equiv \frac{q}{\hbar}\int_{r_0}^{r}\boldsymbol{A}(\boldsymbol{r}')\cdot d\boldsymbol{r}'$.

where r_0 is some arbitrarily chosen reference point. Using this definition, show that the Schrödinger equation for the new wavefunction Ψ' reduces to a usual Schrödinger equation without the vector potential $\boldsymbol{A}$.

(d) Back to the experiment. Before exiting the entrance slits, the beam is moving through a region with $\boldsymbol{A} = \boldsymbol{0}$. Using the vector potential found in (a), show that the wavefunctions acquire a phase $g = \pm\frac{q}{2\hbar}\Phi$, resulting in a (measurable) phase difference of $\frac{q}{\hbar}\Phi$, as electrons pass the solenoid on opposite sides although the **B** field itself is zero.

(e) Based on these observations, does the vector potential have a physical meaning and is the quantum theory gauge invariant?



Problem 5:

(a) Derive the expression and calculate the capacitance per unit length for a piece of coaxial cable in SI units. The coaxial cable has a diameter of an inner wire of 1 mm and an inner diameter (ID) of an outer shield of 5 mm. The dielectric constant of the insulator between the two conductors is $\kappa_e = 1.5$. The dielectric permeability of the free space is $\varepsilon_0 = 8.8*10^{-12}$ F/m.

(b) Derive the expression and calculate the inductance for this cable in SI units, assuming that the relative magnetic permeability is unity $\kappa_m = 1$. The magnetic permeability of the free space is $\mu_0 = 4\pi*10^{-7}$ H/m. For problems (a) and (b) assume that the skin depth of the wire and shield are zero (a high-frequency limit).

(c) Derive the expression and calculate the wave impedance for this cable.

Problem 6:

The particle accelerator accelerates electrons to a speed of 0.999 999 999 7 c, which is very nearly equal to the speed of light.

(a) What is the rest energy (in MeV) of the electrons?

(b) Find the magnitude of the relativistic momentum of the electrons. Comparing it with the non-relativistic value, is it bigger, smaller? By what factor?

(c) Compute the electron's total energy (in GeV).

(d) Assume the electrons collide with the nuclei of "stationary" hydrogen atoms in a gas cell. Assume the collision products are a proton, an electron, and a particle-antiparticle pair. What is the rest energy of the most massive particle (or antiparticle) that can be created in such a collision?

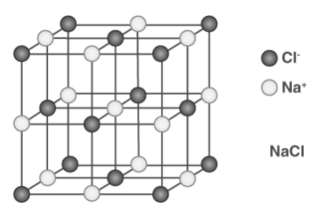
Problem 7:

Bonding is essential to the structure of solids.

(a) Describe the different kinds of bonds in solids, their origins, and general character.

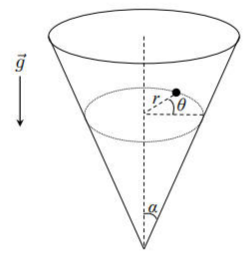
(b) What kind of bonding applies to NaCl, Si, H_20 ice, Na, and Ar?

(c) For NaCl crystals the ions of Na and Cl are on the cubic lattice shown in the figure. If the density of NaCl is 2.16 g/cm^3, what are the distances between nearest, next-nearest, and next-next-nearest neighboring atoms?
(Na atomic mass is 22.99g/mole and Cl atomic mass is 35.45 g/mole, Avogardro's constant is $N_A = 6.022x10^{23}mol^{-1}$).



Problem 8:

A particle of mass m is moving on the surface of a cone placed vertically with the vertex downwards in a uniform gravitational field. The vertical angle of the cone is 2α .



(a) Obtain the Lagrange equations of motion.

(b) What quantities are conserved? What is the energy of the system?

(c) Determine the number of physical turning points in the orbit. That is, does an infalling particle plunge, bounce to infinity, or oscillate back and forth in distance? Show your work.