August 2016 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Trigonometric identities: $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Solve 6 out of the 8 problems! (All problems carry the same weight)

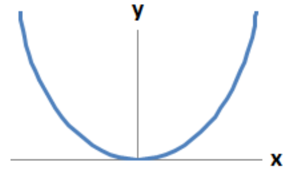
Problem 1:

A bead, of mass m, slides without friction on a wire that is in the shape of a cycloid with equations

$x = a(2\theta + \sin 2\theta)$,

$y = a(1 - \cos 2\theta)$,

$-\pi/2 \le \theta \le \pi/2$.



A uniform gravitational field **g** points in the negative y-direction.

(a) Find the Lagrangian and the second order differential equation of motion for the coordinate θ.

(b) The bead moves on a trajectory s with elements of arc length ds.

Integrate $ds = (dx^2 + dy^2)^{½} = ((dx/d\theta)^2 + (dy/d\theta)^2)^{½} d\theta$ with the condition $s = 0$ at $\theta = 0$ to find s as a function of θ

(c) Rewrite the equation of motion, switching from the coordinate θ to the coordinate s and solve it. Describe the motion.

Problem 2:

Nuclear transitions have very narrow linewidths, so gamma emission and re-absorption by another atom does not usually occur, as the photon carries momentum and there is loss of energy with recoil of the atoms. However, in the Mossbauer Effect (discovered 1957, Nobel prize 1961), a gamma photon of energy E emitted from the nucleus of an atom can be absorbed by another if the two atoms are bound within crystals so that they do not recoil with emission and absorption. The Mossbauer Effect and the very narrow linewidths of nuclear transitions enable measurement of very small changes in gamma ray frequency, on the order of one part in 10^{15}.

In 1960, Einstein's theory of general relativity was experimentally tested with help from the Mossbauer Effect.

If the emitting atom/crystal is a height H above the absorber, the photon gains energy $\Delta E = (gH/c^2)E$ as it "falls" towards the absorber. (Here, g is the acceleration due to gravity and c is the speed of light.) Consequently, by making use of the Doppler effect, re-absorption occurs if the emitting atom/crystal is moving away from the absorber at a speed v.

(a) Find an expression for v as a function of height H.

(b) By making a valid approximation that may be needed, calculate the small non-zero value for v if $H = 22.6$ m and $g = 9.8$ m/s^2.

[In the original experiment, H = 74 ft = 22.6 m.]

Problem 3:

A thin disc with radius R and uniform charge density σ is centered at the origin. Its normal points along the z-axis. It is rotating with angular velocity ω **k** about the z-axis.

(a) Find the potential and the electric field due to the disc on the z-axis.

(b) Show that the potential reduces to that of a point charge for large distances from the origin.

(c) Find the magnetic moment of the disk.

(d) At large distances from the origin, find the magnitude and direction of the Poynting vector **S**. Does the rotating disk produce a radiation field? Explain.

Problem 4:

Consider a quantum system with just three linearly independent states.
Suppose the Hamiltonian, in matrix form, is

$$\mathrm{H} = \mathrm{V}_0 \begin{pmatrix} 1-\epsilon & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & \epsilon & 2 \end{pmatrix},$$

where V_0 is a constant and $\varepsilon << 1$.

(a) Write down the eigenvalues and eigenvectors of the unperturbed Hamiltonian, H^0 ($\varepsilon = 0$).

(b) Solve for the exact eigenvalues of H.
Expand each of them in a power series in ε up to second order.

(c) Use first- and second-order non-degenerate perturbation theory to find the approximate eigenvalue for the state that grows out of the non-degenerate eigenvector of H^0.

(d) Use degenerate perturbation theory to find the first-order corrections to the initially degenerate eigenvalues.

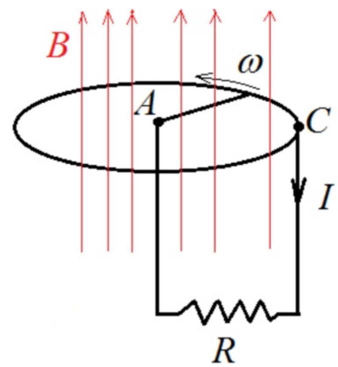
Problem 5:

Refer to the figure. One end of a conducting rod rotates with angular velocity ω in a circle of radius a making contact with a horizontal, conducting ring of the same radius. The other end of the rod is fixed. Stationary conducting wires connect the fixed end of the rod (A) and a fixed point on the ring (C) to either end of a resistance R. A uniform vertical magnetic field **B** passes through the ring.

(a) Find the current I flowing through the resistor and the rate at which heat is generated in the resistor,

(b) What is sign of the current if positive I corresponds to flow in the direction of the arrow in the figure?

(c) What torque must be applied to the rod to maintain its rotation at the constant angular rate ω?
What is the rate at which mechanical work must be done?



Problem 6:

Consider a beam of N silver atoms per second in their ground state. The atoms are polarized in the |+> state, (i.e. the spin-up state of S_z), and travel along the y-axis with a constant velocity of magnitude v_0.
The atoms traverse a region of space of length L which contains a uniform static magnetic field of strength B, directed along the y-axis, i.e. the direction of travel.
Note: The translational motion of the center of mass of the atoms is treated classically.

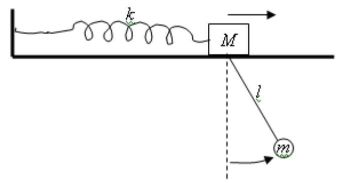
(a) Upon leaving the region of length L, the atoms enter a spin analyzer (a Stern-Gerlach device) with its magnetic field directed along the positive z-axis and the field gradient pointing in the –z direction. What is the number of atoms per second in either of the beams emerging from the analyzer?

(b) How should the analyzer be oriented, (i.e. what should be the direction of the magnetic field of the analyzer), so that only one beam emerges from the analyzer? Interpret this result physically.

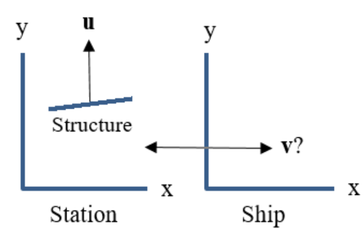
Problem 7:
A pendulum consisting of a mass m and a weightless string of length ℓ is mounted on a mass M, which in turn slides on a support without friction and is attached to a horizontal spring with force constant k, as seen in the diagram. There is a slot in the support in order that the pendulum may swing freely.

(a) Set up Lagrange's equations.

(b) Find the normal mode frequencies for small oscillations. What are those frequencies to zeroth order in m/M, when m << M?



Problem 8:
A spaceship travels with velocity **v** = v**i** with respect to a space station. In the frame of the space station, a linear structure of length L = 10 km moves with velocity **u** = (c/2)**j** in the positive y direction. The structure lies in the xy-plane and makes an angle θ = 7.2° with the x axis? When viewed from the spaceship, the structure is aligned with the x-axis.



(a) What is the velocity **v** of the spaceship (magnitude and direction)?

(b) What is the length of the structure in the frame of the spaceship'?