August 2018 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $q_e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

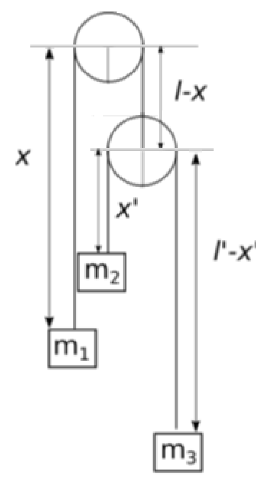
Problem 1:

Consider the matrix of the operator A, $\mathbf{A} = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix}$.

(a) Find the matrix sin ($\mathbf{A}$).
(b) Find the matrix $\cos(\mathbf{A} - (\pi/2)\mathbf{I})$, where $\mathbf{I}$ is the identity matrix.
(c) Find the matrix $\exp(\mathbf{A}^n)$, for arbitrary n = 0, 1, 2, ... (Hint: even and odd n are different.)

Problem 2:

A mechanical system known as an Atwood machine consists of three weights of mass m_1, m_2, and m_3, respectively, connected by a light (massless) inextensible cords of length $l - \pi R$ and $l' - \pi R$, respectively, which pass over identical pulleys with radius R, mass M, and moment of inertia I. Find the acceleration of m_1.
Let $m_1 = m$, $m_2 = m$, $m_3 = 0.25$ m,
M = 0.5 m and $I = 0.25\ mR^2$.



Problem 3:

A cylindrical solenoid L = 50 cm long with a radius of r = 3 mm has N = 500 tightly-wound turns of low-resistance wire uniformly distributed along its length. The solenoid is connected, using low-resistance wires, in series with a R_S = 20 ohm resistor, a V = 9-volt battery, and a switch, which is initially open.

Around the middle of the solenoid is a two-turn rectangular loop l = 3 cm by w = 2 cm made of resistive wire having a total resistance of R_L = 150 ohms. The plane of the loop is perpendicular to the axis of the solenoid and the centers of the loop and solenoid coincide. The switch is closed at t = 0.

Showing all your work, develop an expression, in terms of the above symbolic quantities, for the current as a function of time in the rectangular loop, and evaluate that current at t = 1 microsecond after the switch is closed.

Problem 4:

A wire is bent into the shape given by $y = A|x^n|$, where $n \geq 2$, and oriented vertically, opening upward, in a uniform gravitational field **g**. The wire rotates at a constant angular velocity $\boldsymbol{\omega}$ about the y-axis, and a bead of mass m is free to slide on it without friction.

(a) Find the equilibrium height of the bead on the wire.

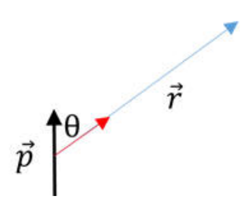
(b) Consider especially the case n = 2. Find the frequency of small vibrations about the equilibrium position.

Problem 5:

For a harmonically oscillating dipole $\mathbf{p} = \mathbf{p}_0 \sin(\omega t)$ the electric and the magnetic field vectors in in the radiation zone are given by

$\mathbf{E}_R(\mathbf{r},t) = -(1/(4\pi\varepsilon_0 c^2 r))(d^2\mathbf{p}_\perp(t - r/c)/dt^2)$,

$\mathbf{B}_R(\mathbf{r},t) = (\mathbf{r}/(rc))\times\mathbf{E}_R(\mathbf{r},t)$.

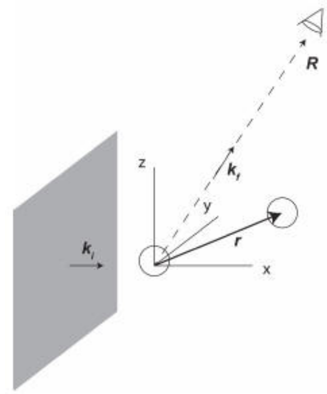


(a) Find the time-average power radiated per unit solid angle as a function of the angle θ (as defined on the diagram above).

(b) Sketch this distribution as function of θ. In the same plot indicate also the electric and magnetic field vectors for some selected point in space (draw directions of these vectors).

(c) Find the time-average power radiated to the whole space. How this power depends on frequency ω of the oscillations?

Problem 6:



A plane wave with wave vector $\mathbf{k}_i$ parallel to the x-axis (see figure) illuminates two atoms that radiate spherical wave fronts. One atom is located $\mathbf{r}$ (a vector) away from the reference atom at the origin. An observer is located very far away at $\mathbf{R}$ (another vector). Assume the scattering process is elastic, i.e., the energies of the atoms are not affected by the scattering.

(a) Write the scattered wave function, ψ_s, in terms of $\mathbf{k}_f$, $\mathbf{r}$ and $\mathbf{R}$. Assume the incident (plane) wave function is $\psi_i = \psi_0 \exp(i(k_i x - \omega t))$.

(b) The intensity, a real number, of the radiation at $\mathbf{R}$ is the square modulus of the wave function. Write an expression for the intensity of the scattered wave function at $\mathbf{R}$.

(c) What is the condition for $\mathbf{k}_f \cdot \mathbf{r}$ such that the scattering is maximum (constructive interference)?

(d) What is the condition for $\mathbf{k}_f \cdot \mathbf{r}$ such that the scattering is minimum (equal to zero, destructive interference)?

(e) Imagine a third atom is inserted halfway between the first two. How are the conditions for maximum scattering (constructive interference) and minimum scattering (destructive interference) changed?

Problem 7:

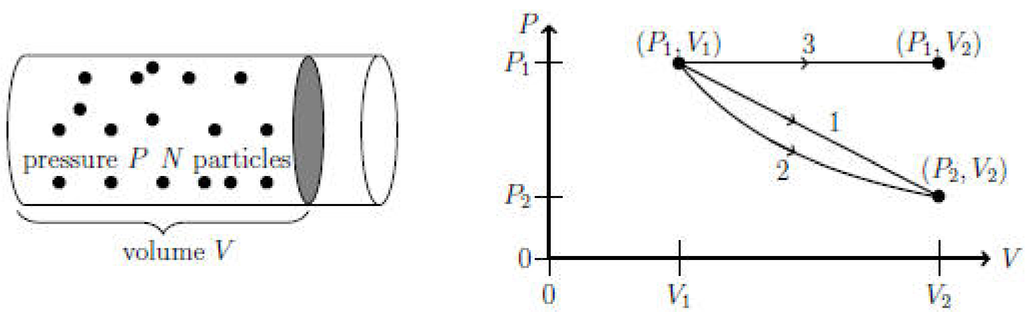
(a) Particles are incident on a spherically symmetric potential energy function $U(r) = (\beta/r)\exp(-\gamma r)$, where β and γ are constants. Show that in the Born approximation the differential scattering cross-section for the scattering vector $\boldsymbol{\kappa}$ is given by

$$\sigma_k^B(\theta,\varphi) = d\sigma_k^B/d\Omega = [2m\beta/(\hbar^2(\kappa^2 + \gamma^2))]^2.$$

(b) Use this result to derive the Rutherford formula for the scattering of α-particles, namely that, for α-particles of energy E incident on a stationary nucleus of atomic number Z, the differential scattering cross-section for scattering at an angle θ to the incident direction is

$$d\sigma/d\Omega = [Ze^2/(2E\sin^2(\theta/2))]^2.$$

Problem 8:



Consider an ideal gas of N particles in a cylinder with a piston so that the volume and pressure may change. Suppose that each particle has $f = 5$ quadratic degrees of freedom in its energy. Consider the three paths in a PV diagram shown. Suppose ways 1 and 3 are straight lines on the PV diagram and way 2 is an adiabatic process.

(a) How much work W is done on the gas for ways 1, 2, and 3?
(b) How much heat Q is transferred to the gas for ways 1, 2, and 3?
(c) What is the change in internal energy ΔU for ways 1, 2, 3?
(d) What is the change in entropy ΔS for way 3?