August 2013 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider the decay Λ^0 --> n + π^0, followed by π^0 --> 2γ.
(a) Given the masses M_Λ, M_n and M_π, find the energy of the decay products n and π^0 in the rest frame of the Λ^0.
(b) The two gamma rays from the decay of the π^0 are observed to have equal energies in the rest frame of the Λ^0. Find the angle between the two gamma rays in this frame, in terms of the particle masses.

Problem 2:

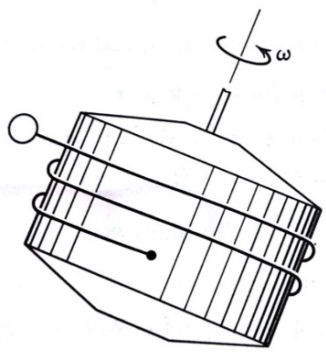
A particle is represented (at time t = 0) by the wave function

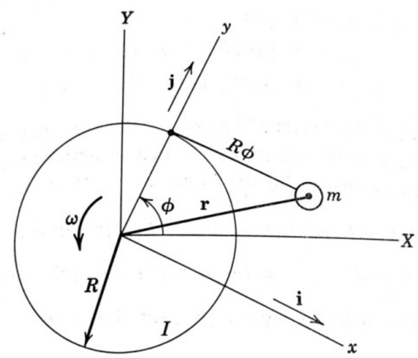
$$\Psi(x,0) = \begin{cases} A(a^2 - x^2) & \text{if } -a \leq x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.
(b) What is the expectation value of x (at time t = 0)?
(c) What is the expectation value of p (at time t = 0)?
(d) Find the expectation value of x^2.
(e) Find the expectation value of p^2.
(f) Find the uncertainty in x (Δx).
(g) Find the uncertainty in p (Δp).
(h) Check that your results are consistent with the uncertainty principle.

Problem 3:

A simple device was used in the Pioneer III lunar probe, which was capable to reduce its spin to zero. The device consists of a small mass m on the end of a light cord wrapped around the symmetrical spinning body. With the satellite spinning with angular speed ω_0 about its axis of symmetry, the mass m is released. The cord will unwind and the angular speed of the satellite will decrease. When the cord is completely unwound, it is released and allowed to fly away. By choosing the length of the cord properly, the spin of the satellite can be reduced to any value less than the initial value.





(a) Let R, ω, ϕ be defined in the figure above. Assume that the mass m is released when $\phi = 0$. Show that the velocity of the mass may be written as
$\mathbf{v} = -R\omega\, \mathbf{i} + (\omega + d\phi/dt)R\phi\, \mathbf{j}$,
where **i** and **j** be unit vectors pointing along the x- and y-axis, respectively.

(b) In terms of R, ω, ϕ, $d\phi/dt$, and I, the moment of inertia of the satellite, find expressions for the kinetic energy and the angular momentum of the system.

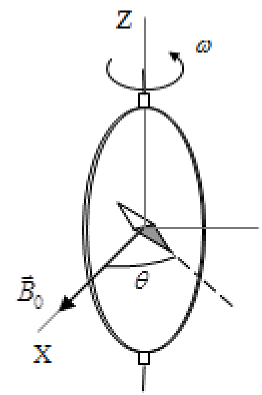
(c) Use conservation of energy and angular momentum to show that
$C(\omega_0^2 - \omega^2) = \phi^2(\omega + d\phi/dt)^2$ and $C(\omega_0 - \omega) = \phi^2(\omega + d\phi/dt)$, and find the constant C.

(d) Solve for $d\phi/dt$ and ω in terms of ω_0 and C.

(e) Find the length of the cord required for a mass m to spin a satellite with moment of inertia I and radius R down to zero.

Problem 4:

(a) A closed circular coil of N turns, radius a and total resistance R is rotated with uniform angular velocity ω about a vertical diameter in a horizontal magnetic field $\mathbf{B}_0 = B_0\mathbf{i}$. Compute the emf ε induced in the coil, and also the mean power <P> required for maintaining the coil's motion. Neglect the coil self inductance.



(b) A small magnetic needle is placed at the center of the coil, as shown in the figure. It is free to turn slowly around the z-axis in a horizontal plane, but it cannot follow the rapid rotation of the coil. Once the stationary regime is reached, the needle will point in a direction making a small angle θ with $\mathbf{B}_0$. Compute the resistance R of the coil in terms of this angle and the other parameters of the system. (Lord Kelvin used this method in the 1860s to set the absolute standard for the ohm.)

Useful integrals:

$$\int_0^{2\pi} \sin x\, dx = \int_0^{2\pi} \cos x\, dx = \int_0^{2\pi} \sin x \cos x\, dx = 0\,, \quad \int_0^{2\pi} \sin^2 x\, dx = \int_0^{2\pi} \cos^2 x\, dx = \pi\,, \quad \text{and later} \quad \int x^n\, dx = \frac{1}{n+1} x^{n+1}$$

Problem 5:

A coherent state $|\lambda\rangle$ for a simple harmonic oscillator with frequency ω is an eigenstate of the lowering operator: $a|\lambda\rangle = \lambda|\lambda\rangle$. (Note that λ can be a complex number.)
A coherent state is not an eigenstate of the Hamiltonian H, but can be expanded in terms of the eigenstates of H, $|\lambda\rangle = \sum_0^\infty b_n|n\rangle$,
(a) Show that, up to an overall normalization, a coherent state can be expressed as $|\lambda\rangle = \exp(\lambda a^\dagger)|0\rangle$. Here $a^\dagger$ is the raising operator and $|0\rangle$ is the ground state.

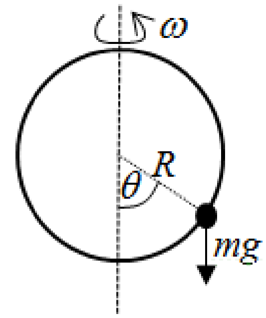
(Hint: For any operator A, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \cdots$.)

(b) Start with a coherent state $|\lambda_0\rangle$ at time t = 0. Show that up to an overall phase, under time evolution this state evolves into a coherent state $|\lambda(t)\rangle$. Express $\lambda(t)$ in terms of λ_0.
(c) Calculate time dependent expectation values of coordinate and momentum for a simple harmonic oscillator in a coherent state. Show that these quantities evolve as the classical coordinate and momentum of a harmonic oscillator. Start with a coherent state $|\lambda_0\rangle$ at time t = 0 taking λ_0 to be a real number.

Useful relations:
$a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle$, $x = (\hbar/(2m\omega))^{1/2}\langle a^\dagger + a\rangle$, $p = i(m\omega\hbar/2)^{1/2}\langle a^\dagger - a\rangle$

Problem 6:

A bead of mass m can slide without friction on a circular loop of radius R. The loop lies in a vertical plane and rotates about a vertical axis with constant angular velocity ω, as shown in the figure.

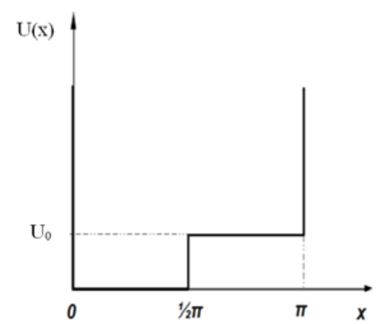


(a) For angular velocity ω greater than some critical angular velocity ω_c, the bead can undergo small oscillations about some stable equilibrium point θ_0. Find ω_c and θ_0.
(b) Obtain the equations of motion for the small oscillations about θ_0 as a function of ω and find the period of oscillations

Problem 7:

A particle is moving in a one dimensional potential,

$$U(x) = \begin{cases} \infty & \text{for } x < 0 \text{ and } x > \pi, \\ 0 & \text{for } 0 \le x \le \pi/2, \\ U_0 > 0 & \text{for } \pi/2 < x \le \pi. \end{cases}$$



(a) The variational method can be used to find an upper bound for the ground-state energy of the particle. For the trial function $\Psi = N(\pi/2)^{1/2}(\sin(x) + a\sin(2x))$, find the optimal value of the variational parameter a.
(b) For $U_0 = 0.1*\hbar^2/(2m)$, estimate the ground state energy of the particle in terms of $\hbar^2/(2m)$.
(c) Find the wave function and energy of the ground state for the cases where $U_0 \to 0$ and $U_0 \to \infty$.

Problem 8:

In reference frame K a long, straight, neutral wire (charge density $\rho = 0$) with a circular cross sectional area $A = \pi r^2$ lies centered on the z-axis and carries a current with uniform current density J **k**.
(a) Find the scalar potential Φ, the vector potential **A**, the electric field **E**, and the magnetic field **B** at a point P on the x-axis a distance $x > r$ from the wire.
(b) In a frame K' moving with velocity v **k** with respect to K, find the ρ, **J**, Φ, **A**, **E**, and **B** at the point P.
(c) In a frame K" moving with velocity v **i** with respect to K, find the ρ, **J**, Φ, **A**, **E**, and **B** at the point P.