August 2014 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\,\lambda)) - 1]^{-1}$

Maxwell-Boltzmann speed distribution:

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}}$$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

The Moon orbits the Earth in nearly circular orbit with a radius of ~400,000 km in 29.5 days. Imagine a second moon, with a mass 1/8th that of the Moon, orbiting in the same circular orbit but the opposite direction. At time $t = 0$, these two moons collide and fuse into a single moon whose mass is the combined mass of the individual moons.

(a) What is the perigee (distance of closest approach to the Earth) of the merged moon's orbit, assuming one can neglect the mass of the moons relative to the Earth?

(b) What is the orbital period of the merged moon?

Problem 2:

A thin spherical shell of radius a has a surface potential $\Phi(a,\theta) = \Phi_0 \cos\theta$, where θ is the usual polar angle relative to the z-axis, and Φ_0 is a constant.

(a) Find the electric potential $\Phi(r,\theta)$ everywhere outside the shell. Calculate the equivalent electric dipole moment **p** that produces this potential Φ.

(b) Assume that previously calculated dipole moment acquires a $\cos(\omega t)$ time dependence. Calculate the magnetic field **B** and the electric field **E** in the radiation zone.

(c) Find the time-averaged power $d\langle P\rangle/d\Omega$ radiated per unit solid angle. Express your result in terms of ω, θ, a, Φ_0, ε_0 and c.

(d) Find the total power $\langle P\rangle$ radiated.

Problem 3:

Suppose a small meteorite makes a hole of area $A = 1\ mm^2$ in the wall of a spaceship. The habitable volume of the spaceship is $V = 10\ m^3$. The temperature of the air in the spaceship is $T = 27^o$ C and the pressure $P = 105$ kPa. The molar mass of air is $M = 29$ g /mole. Estimate, how much time will be available to astronauts to put on spacesuit, if the pressure should not drop by more than to 50% of its initial value.

Problem 4:

Consider a one-dimensional crystal with primitive lattice translation a. Let {|n>} be a set orthonormal electron states, n = $-\infty$ to $+\infty$. Assume that in the subspace spanned by {|n>} the matrix elements of the electron Hamiltonian are given by
$\langle n|H|n\rangle = E_0$, $\langle n|H|n\pm1\rangle = -A$, $\langle n|H|n\pm2\rangle = B$, $\langle n|H|m\rangle = 0$ for all other m, with
$E_0 = (7/8)\hbar^2/(ma^2)$, $A = (1/2)\hbar^2/(ma^2)$, and $B = (1/16)\hbar^2/(ma^2)$.

(a) Assume the eigenstates of H are of the form $|\Phi\rangle = \sum_n b(x_n)|n\rangle$. Write down the coupled linear equations for the $b(x_n)$. Note that $b(x_{n\pm1}) = b(x_n \pm a)$.

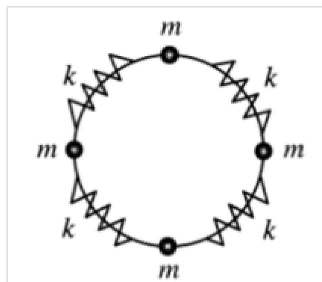
(b) Try solutions of the form $b(x_n) = \exp(ikx_n)$ and show that E as a function of k is given by

$$E(k) = \frac{\hbar^2}{ma^2}\left(\frac{7}{8} - \cos ka + \frac{1}{8}\cos 2ka\right).$$

(c) Determine the effective mass at the bottom of the energy band and at the top of the band from a quadratic expansion of E in the departure of k from these points.

Problem 5:

Consider a ring of radius, r, with 4 identical point particles of mass m interconnected by identical springs with spring constant k to their nearest neighbors. The particles move without friction.
The interconnected springs can be viewed as causing harmonic oscillations.



(a) Determine the number of normal modes of oscillations, and establish the Lagrangian.

(b) Find the frequencies for small oscillations and describe the corresponding eigen-vibrations.

Problem 6:

Consider an infinite plane with a uniform charge density σ located at z = 0.

(a) Using Gauss' law, find the electric field created by this plane.

(b) Find the potential Φ(z).

(c) Locate another plane with charge density -σ at z = d. Find the potential Φ(z) everywhere. What is the magnitude of thepotential jump across the dipolar layer configurations of the two planes?

(d) Find the force per unit area between the planes.

Problem 7:

A particle moves in a horizontal plane on the surface of the Earth. Show that the magnitude of the horizontal component of the Coriolis force is independent of the direction of the motion of the particle.

Problem 8:

Consider a system of two non-identical spin ½ particles. For $t < 0$ the particles do not interact and the Hamiltonian may be taken to be zero. For $t > 0$ the Hamiltonian is given by

$$H = (4\Delta/\hbar^2)\, \mathbf{S}_1 \cdot \mathbf{S}_2,$$

where Δ is a constant. For $t < 0$ the state of the system is $|+ -\rangle$.

(a) For $t > 0$, find, as a function of time, the probability that for finding the system in each of the states $|+ +\rangle$, $|+ -\rangle$, $|- +\rangle$, and $|- -\rangle$, by solving the problem exactly.

(b) For $t > 0$, find, as a function of time, the probability that for finding the system in each of the states $|+ +\rangle$, $|+ -\rangle$, $|- +\rangle$, and $|- -\rangle$, using first-order time dependent perturbation theory with H a perturbation that is switched on at $t = 0$.

(c) Under what condition is the perturbation calculation a bad approximation to the exact solution, and why?