# August 2015 Qualifying Exam

# Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

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**Physical Constants:**

**Planck constant:** $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
**Boltzmann constant:** $k_B = 1.3806504 * 10^{-23}$ J/K
**Elementary charge:** $e = 1.602176487 * 10^{-19}$ C
**Avogadro number:** $N_A = 6.02214179 * 10^{23}$ particles/mol
**Speed of light:** $c = 2.99792458 * 10^{8}$ m/s
**Electron rest mass:** $m_e = 9.10938215 * 10^{-31}$ kg
**Proton rest mass:** $m_p = 1.672621637 * 10^{-27}$ kg
**Neutron rest mass:** $m_n = 1.674927211 * 10^{-27}$ kg
**Bohr radius:** $a_0 = 5.2917720859 * 10^{-11}$ m
**Compton wavelength of the electron:** $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
**Permeability of free space:** $\mu_0 = 4\pi\ 10^{-7}$ N/A$^2$
**Permittivity of free space**: $\varepsilon_0 = 1/\mu_0 c^2$
**Gravitational constant:** $G = 6.67428 * 10^{-11}$ m$^3$/(kg s$^2$)
**Stefan-Boltzmann constant:** $\sigma = 5.670\ 400 * 10^{-8}$ W m$^{-2}$ K$^{-4}$
**Wien displacement law constant:** $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
**Plank radiation law:** $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

**Solve 6 out of the 8 problems!** (All problems carry the same weight)

**Problem 1:**
Consider a single electron moving with an anharmonic potential energy given by $U(x) = \frac{1}{2} kx^2 + k'x^4$. Let $H_0$ be the Hamiltonian of the system when $k' = 0$.
(a) Use perturbation theory to find the energies of the ground and excited states to first order, assuming that $k' << k$.
(b) Expand the ground state to first order in terms of the eigenstates $\{|n\rangle\}$ of $H_0$.

**Problem 2:**
A particle of mass m is subjected to a force whose potential energy is $U(x) = ax^2 - bx^3$, with a and b constants and $a > 0$.
(a) Find the force.
(b) Assume that the particle starts at the origin with velocity of magnitude $v_0$. Show that if $v_0 < v_C$, where $v_C$ is a certain critical velocity, the particle will be confined in a region near the origin. Find $v_C$.

**Problem 3:**
A quantum system with two states has the Hamiltonian matrix ($\varepsilon_1 < \varepsilon_2$)

$$H = \begin{pmatrix} \varepsilon_1 & v \\ v^* & \varepsilon_2 \end{pmatrix}.$$

(a) What are the two energies $E_\pm$ of the system?
(b) It is of advantage to parameterize the eigenstates as

$$\psi = \begin{pmatrix} \cos\alpha \; e^{i\varphi} \\ \sin\alpha \; e^{-i\varphi} \end{pmatrix}$$

with real $\alpha$, $\varphi$. Show that this state is normalized.
Show that $\varphi = \gamma/2$ for complex off-diagonal matrix element $v = |v|e^{i\gamma}$, and find the values of $\alpha$ for the two eigenstates.

**Problem 4:**

A plasma is generated inside a long hollow cylinder of radius R. It has the charge distribution

$$\rho(r) = \frac{\rho_0}{[1+(r/a)^2]^2} \ ,$$

where r is the distance to the center, and $\rho_0$ and a are constants.

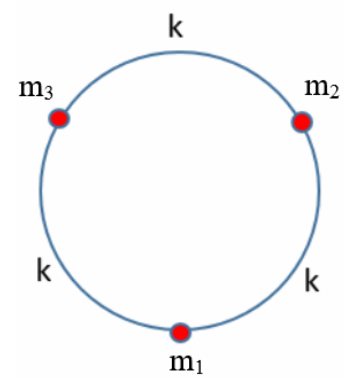
(a) What is the electric field inside and outside the cylinder?

(b) Setting $V(r=0) = 0$, find the potential at all points $r < R$.

(c) What are the equilibrium positions of a particle with charge q placed inside the cylinder, assuming the charge does not alter $\rho(r)$. What is the force acting on the particle if it is displaced by a distance $\varepsilon << a$ from an equilibrium position. Are the equilibrium positions stable?

**Problem 5:**

Three point masses of mass m move on a circle of radius R. The equilibrium positions are shown in the figure. Each point mass is coupled to its two neighboring points by a spring with spring constant k.




(a) Write down the Lagrangian of the system.

(b) Find the normal modes of the system.

(c) If at t = 0 mass 1 is displaced from its equilibrium position clockwise by $9^o$, what is the subsequent motion of all the masses?

**Problem 6:**

Consider N >> 1 non-interacting spin-½ particles with mass M confined to a cubical box of volume V.

(a) Derive an expression for the Fermi energy.

(b) Make some numerical estimates, assuming in each case that the particles in question are non-interacting, for the Fermi energy of

(i) electrons in a typical metal,

(ii) nucleons in a large nucleus,

(iii) $^3$He atoms in liquid $^3$He, which has an atomic volume of about 0.05 $nm^3$ per atom.

**Problem 7:**
Twins Alice and Bob, who are 19 years old, leave the earth and travel to a distant planet 12 light-years away. Assume that the planet and earth are at rest with respect to each other. The twins depart at the same time on different spaceships. Alice travels at a speed of 0.5c, and Bob travels at 0.9c.
(a) What is the difference between their ages when they meet again on earth at the earliest possible time, and which twin is older?
(b) If instead of traveling at a constant speed of 0.5c, Alice had covered the total distance in the same time, but on her trip to and from the planet had accelerated and travelled 50% of the time with speed v and 50% of the time with speed of 3v, as measured on earth, how old would she be when she meets Bob again.

**Problem 8:**
Consider a hydrogenic atom with nuclear charge $Zq_e$ in a **strong** magnetic field $\mathbf{B} = B\mathbf{k}$ Assume that the Zeeman slitting is much larger than the spin-orbit splitting of the energy levels, so that to first order the spin-orbit interaction can be ignored.
(a) Find the energies of the 2s and 2p energy levels in the strong magnetic field. Is the degeneracy completely removed by the Zeeman interaction?
(b) Estimate the magnitude of the magnetic field, B, required to give a Zeeman splitting in the hydrogenic atom comparable to the binding energy of the ground state of the hydrogen atom. Can such a magnetic field be created in the laboratory?