Spring 2012 Qualifying Exam

Part II

Mathematical tables are provided. Formula sheets are provided.

Calculators are allowed.

Please mark exactly six attempted problems for grading.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma =$ 5.670 400 * 10^{-8} W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

A small child (assume a point child!) of mass m sits in a massless swing suspended by a massless rope of length L. Her sister pulls the child back until the rope makes an angle $\theta_i = ½$ radian with the vertical. Rather than releasing the child, the sister pushes with a force of constant magnitude $F = mg$ tangential to the arc of the circle, releasing when $\theta = 0$.

(a) How high up will the child go? What angle will the swing make at its highest point?

(b) How long did the sister push?

Problem 2:

Consider that a photon is scattered by an electron initially at rest (Compton scattering).

(a) Derive an expression for the shift in the wavelength of the photon as a function of its scattering angle θ.

(b) Which photon scattering angle corresponds to the largest Compton shift and why?

(c) At what minimum photon energy can half of the photon energy be transferred onto the electron?

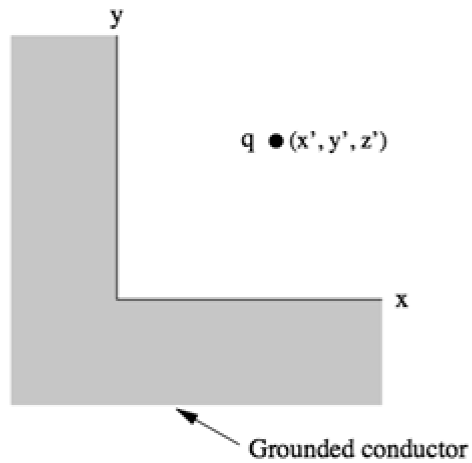
Problem 3:

Show that the wave equation $\partial^2\psi(x,t)/\partial x^2 - (1/c^2)\partial^2\psi(x,t)/\partial t^2 = 0$, with appropriate initial/boundary conditions, is invariant under the Lorentz transformation, i.e., show that $\partial^2\psi/\partial x^2 - (1/c^2)\partial^2\psi /\partial t^2 =\partial^2\psi/\partial x'^2 - (1/c^2)\partial^2\psi /\partial t'^2$.

(Consider only one space component x and the time component t.)

Problem 4:

A point charge q is placed at a position (x',y',z') near a grounded conducting "corner," which occupies the region $x < 0$, $y < 0$, $-\infty < z < \infty$, as shown in the figure below.
(a) Find the electrostatic potential in the region $x > 0$, $y > 0$.
(b) Find the force between the corner and the charge if $x' = y' = a$ and $z' = 0$.

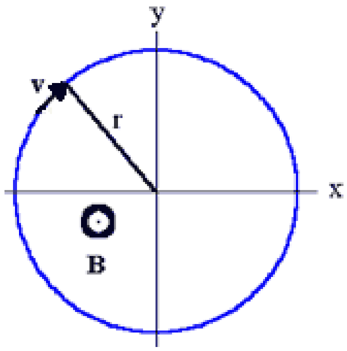


Problem 5:

Two identical spin-zero bosons are placed in a one-dimensional square potential well with infinite high walls.
$V(x) = 0$, $0 < x < L$, $V(x)$ infinite otherwise.
(a) Assume the bosons do not interact with each other. Write down the wave functions and the energies for the ground state and the first two excited states of the system.
(b) Now assume the bosons interact with each other through the perturbative potential
$H'(x_1, x_2) = -LV_0\delta(x_1 - x_2)$.
Calculate the first order perturbation to the ground-state energy of the system.

Problem 6:

A uniform magnetic field $\mathbf{B} = B_0\mathbf{k}$ points in the z-direction. A particle with mass m and charge q moves with kinetic energy E_0 in the x-y plane in a circular orbit centered at the origin as shown.



At t = 0 the magnetic field strength starts changing slowly, so that at $t = t_1$ it is $\mathbf{B} = B_1\mathbf{k}$. Neglect radiation.

(a) What is the radius of the orbit R_0 of the particle for $t < 0$ in terms of B_0, and E_0?

(b) Assuming that the radius R of the orbit of the particle does not change appreciably while the particle completes one revolution, what is the kinetic energy E_1 of the particle at time t_1 in terms of B_0, B_1, and E_0?

Now assume that the magnetic field stays constant ($\mathbf{B} = B_0\mathbf{k}$), but that the particle is subject to a drag force $\mathbf{F}_d = -m\mathbf{v}/\tau$, where τ is a constant. At time t = 0 the position and velocity of the particle are $\mathbf{R}_0 = (0, R_0, 0)$ and $\mathbf{v}_0 = (v_0, 0, 0)$.

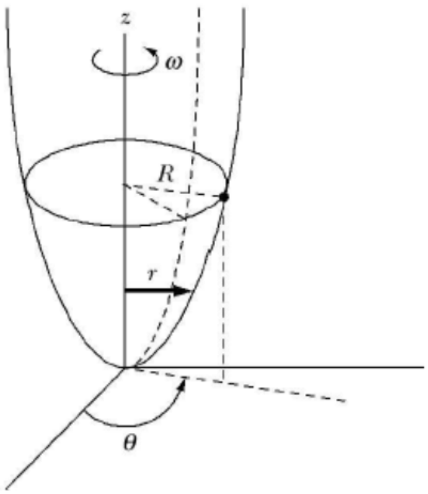
(c) Write Newton's equations of motion for the velocity components v_x, v_y, and v_z.

(d) Construct an equation of motion for $z = v_x + iv_y$, and solve it.

(e) Find expressions for $v_x(t)$, $v_y(t)$, $x(t)$, and $y(t)$. Describe the trajectory of the particle in words.

Problem 7:

A bead of mass m slides along a parabolic wire, where $z = cr^2$. The wire rotates with angular velocity ω about the vertical axis.
(a) Obtain Lagrange's equation for the system.
(b) How fast should the wire rotate in order to suspend the bead at an equilibrium at height $z > 0$?
(c) Assume the particle is displaced by a very small amount from the equilibrium position $r = 0$ and released.
(i) What is the condition for $r = 0$ to be a stable equilibrium position?
(ii) What is the frequency of small oscillation about the equilibrium position?



Problem 8:

The wave function of a system in the ground state is given as

$$\psi(\vec{r},t) = \frac{e^{-i\omega \cdot t}}{\sqrt{\pi \cdot a_0^3}} e^{-\frac{r}{a_0}}$$

(a) Sketch the probability density in coordinate space as function of r/a_0.
(b) Find the momentum space wave function $\bar{\phi}(\vec{p},t)$.
Hint: Use spherical coordinates for evaluation of the integral transform.
(c) Find the probability density function in momentum space. Sketch it as a function of $pa_0/\hbar$.