January 2013 Qualifying Exam

Part II

Mathematical tables are allowed. Formula sheets are provided.

Calculators are allowed.

Please clearly mark the problems you have solved and want to be graded.
Do only mark the required number of problems.

Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\,400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\,7685 * 10^{-3}$ m K

Hydrogenic atom wave functions:

$$R_{10} = 2\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a_0}}, \quad R_{20} = \left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\left(2 - \frac{Zr}{a_0}\right) e^{-\frac{Zr}{2a_0}}, \quad R_{21} = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a}\right)^{\frac{3}{2}} \frac{Zr}{a_0} e^{-\frac{Zr}{2a_0}}.$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}. \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta. \quad Y_{1\pm1} = \mp\sqrt{\frac{3}{4\pi}}\sin\theta e^{\pm i\phi}.$$

Solve 6 out of the 8 problems! (All problems carry the same weight)

Problem 1:

Consider an electron is in the ground state of tritium, for which the nucleus is the isotope of hydrogen with one proton and two neutrons. Suppose that a nuclear reaction instantaneously changes the nucleus into He^3, which consists of two protons and one neutron. Calculate the probability that the electron remains in the ground state of the new atom.

Problem 2:

(a) Consider a circular cylinder of radius R and length L, rotating about its symmetry axis with angular velocity ω and containing an ideal gas with particles of mass M. We assume that the system is in thermal equilibrium at temperature T = 300 K, that the gas is at rest in a reference frame rotating with the cylinder, and that the particle velocities are small enough that we can ignore all the Coriolis forces. If P(0) is the pressure on the axis of rotation, find the pressure P(r) between r = 0 and r = R.

(b) If R = 1 km, the gas consists of Nitrogen molecules, ω = 0.2/s, and P(R) = 101 kPa, find the pressure on the axis.

(c) If R = 0.1 m, the gas consists of Nitrogen molecules, ω = 0.2/s, and P(R) = 101 kPa, find the pressure on the axis.

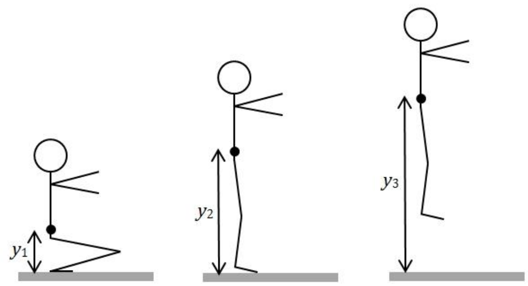
(d) If the cylinder in part (c) suddenly stops spinning, what is the change in the temperature and pressure of the Nitrogen gas? Assume that collisions with the walls of the cylinder can change the momentum, but not the energy of the gas molecules.

Problem 3:

A beam of monochromatic light of wavelength λ in vacuum is incident normally on a nonmagnetic dielectric film of refractive index n and thickness d. Calculate the fraction of the incident energy that is reflected.

Problem 4:

A person of mass m jumps upward from a crouching position, as shown in 3 stages below; the heavy dot represents her center of mass (CM). Her initial crouched position is shown on the left, with her CM a height y_1 above the floor. Her position just as she leaves the ground (lift-off) is shown in the center, with her CM at height y_2. The top of her jump is shown at the right; the maximum height of her CM is y_3. All answers to the questions below must be solely in terms of the given quantities: m, y_1, y_2, y_3, and g.



(a) What is the change in her internal energy, $\Delta E_{internal}$?
(b) What is the speed of her CM at lift-off?
(c) What is the force exerted by the floor on her feet?
(d) How much work is done by the force in (c)?
(e) How long were her feet in contact with the floor while she was actively jumping?

Problem 5:

(a) Consider the momentum four-vector $p_\mu = (p_0, \mathbf{p}) = (E/c, \mathbf{p})$ and show that $(p_0, \mathbf{p})\cdot(p_0, \mathbf{p}) = (mc)^2$ for all inertial reference frames (Lorentz-invariant).

(b) Use this result, along with the conservation of four-momentum, to solve the following problem:
Particle a is pursuing particle b along the x-axis of reference frame S. The two masses are m_a and m_b and the speeds are v_a and v_b with $v_a > v_b$. When a catches up with b they collide and coalesce into a single particle of mass m and speed v. Show that

$$m^2 = m_a^2 + m_b^2 + 2m_a m_b \gamma_a \gamma_b (1 - v_a v_b / c^2)$$

and find v

Problem 6:

Consider the matrix representation of the operator T = $\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$.

(a) Is T Hermitian?

(b) Solve for the eigenvalues. Are they real?

(c) Determine the **normalized** eigenvectors. Since eigenvectors are not unique to within a phase factor, arrange your eigenvectors so that the first component of each is positive and real. Are they orthogonal?

(d) Using the eigenvectors as columns, construct $U^\dagger$, the inverse of the unitary matrix which diagonalizes T. Use this to find this diagonalized version $T^d = U^\dagger T U$

What is special about the diagonal elements?

(e) Compare the determinant |T|, the trace Tr(T), and eigenvalues of T to those of T^d.

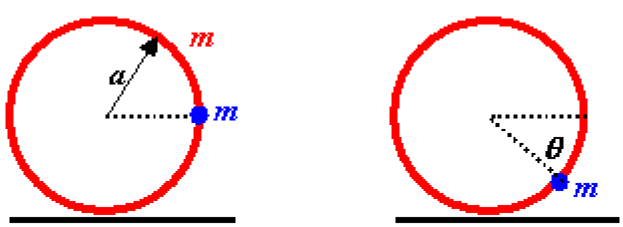
Problem 7:

A particle of mass m is rigidly attached to a uniform hoop of radius a and mass m. The combination is released from rest on a frictionless horizontal surface with the line joining the mass m to the center of the hoop initially horizontal. The plane of the hoop remains vertical throughout the motion.

(a) Write down the Lagrangian of the system.

(b) Find the angular velocity of the hoop as a function of the angle θ.

(b) When θ = π/2, what is the force exerted on the hoop by the surface?



Problem 8:

Consider three reference frames. A meter stick is at rest in frame K_2. It is positioned on the x-axis, from x = 0 to x = 1 m. Frame K_2 moves with velocity $\mathbf{v} = v_2\mathbf{j}$ in the positive y-direction with respect to frame K_1. Frame K_3 moves with velocity $\mathbf{v} = v_3\mathbf{i}$ in the positive x-direction with respect to frame K_1.

(a) Find the velocity of the stick in K_3.

(b) Find the length of the stick in K_3.

(c) Find the angle θ the stick makes with the x-axis in K_3.