January 2015 Qualifying Exam

Part I

Calculators are allowed. No reference material may be used.

Please clearly mark the problems you have solved and want to be graded. Do only mark the required number of problems.

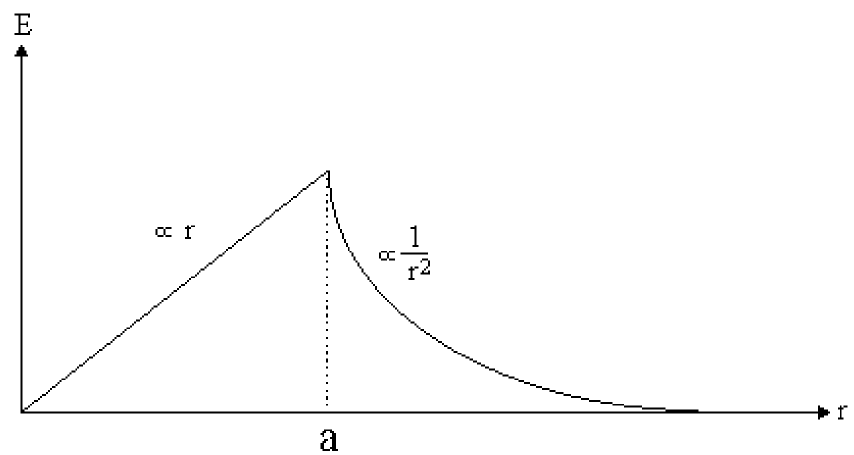
Physical Constants:

Planck constant: $h = 6.62606896 * 10^{-34}$ Js, $\hbar = 1.054571628 * 10^{-34}$ Js
Boltzmann constant: $k_B = 1.3806504 * 10^{-23}$ J/K
Elementary charge: $e = 1.602176487 * 10^{-19}$ C
Avogadro number: $N_A = 6.02214179 * 10^{23}$ particles/mol
Speed of light: $c = 2.99792458 * 10^{8}$ m/s
Electron rest mass: $m_e = 9.10938215 * 10^{-31}$ kg
Proton rest mass: $m_p = 1.672621637 * 10^{-27}$ kg
Neutron rest mass: $m_n = 1.674927211 * 10^{-27}$ kg
Bohr radius: $a_0 = 5.2917720859 * 10^{-11}$ m
Compton wavelength of the electron: $\lambda_c = h/(m_e c) = 2.42631 * 10^{-12}$ m
Permeability of free space: $\mu_0 = 4\pi\ 10^{-7}$ N/A^2
Permittivity of free space: $\varepsilon_0 = 1/\mu_0 c^2$
Gravitational constant: $G = 6.67428 * 10^{-11}$ m^3/(kg s^2)
Stefan-Boltzmann constant: $\sigma = 5.670\ 400 * 10^{-8}$ W m^{-2} K^{-4}
Wien displacement law constant: $\sigma_w = 2.897\ 7685 * 10^{-3}$ m K
Plank radiation law: $I(\lambda,T) = (2hc^2/\lambda^5)[\exp(hc/(kT\ \lambda)) - 1]^{-1}$

Section I:
Work 8 out of 10 problems, problem 1 – problem 10! (8 points each)

Problem 1:
(a) Consider a non-conducting sphere with radius a. This sphere carries a net charge Q, assumed to be uniformly distributed. Find the electric field inside and outside the sphere. Sketch the result.
(b) Now consider a conducting sphere with radius a carrying a net charge Q. Find the electric field inside and outside the sphere.

Solution:
Gauss' law, $\Phi_e = \int_{\text{closed surface}} \mathbf{E}\cdot d\mathbf{A} = Q_{inside}/\varepsilon_0$
(a) Place the center of the sphere at the origin of the coordinate system. Consider a spherical Gaussian surface of radius r centered at the center of the spherical charge distribution.
i. Let r be greater than a, so that the surface encloses the entire charge distribution.
The electric field is radial, the vector **E** is normal to any surface element dA. Thus flux through the surface is $\Phi_e = \int \mathbf{E}\cdot d\mathbf{A} = \int E dA = E\, 4\pi r^2 = Q_{inside}/\varepsilon_0 = Q/\varepsilon_0$
$\mathbf{E} = Q/(4\pi\varepsilon_0 r^2)\, \mathbf{n}$, where $\mathbf{n} = \mathbf{r}/r$.
The field outside the sphere looks like the field of a point charge Q.
ii. Let r be smaller than a, so that the surface only encloses a part of the charge distribution. Now Q_{inside} is the charge density $\rho = Q/V$ times the volume $4\pi r^3/3$ of the distribution which lies inside the spherical Gaussian surface. We therefore have
$\mathbf{E} = \rho r/(3\varepsilon_0)\, \mathbf{n} = Qr/(4\pi\varepsilon_0 a^3)\, \mathbf{n}$.
The field inside the sphere increases linearly with r.



(b) The charge is uniformly distributed over the surface of the conductor.
i. r >a: $\mathbf{E} = Q/(4\pi\varepsilon_0 r^2)\, \mathbf{n}$, r < a: $\mathbf{E} = 0$, the electric field inside a conductor is zero in electrostatics.

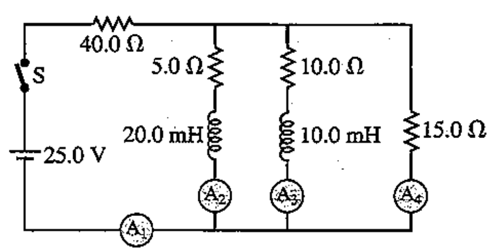
Problem 2:
You have a converging lens with a focal length of 15 cm and an object 5 cm to the left of the lens. How far to the left or right of the lens is the image? Is it upright or inverted? What is its magnification?

Solution
Thin lenses: $1/f = 1/s_o+1/s_i$, $M = -s_1/s_o$
$s_i = 1/(1/f - 1/s_o) = 1/(1/15 - 1/5) = -7.5$ cm
$M = 7.5/5 = 1.5$
The image is 7.5 cm to the left of the lens, upright, and with a magnification of 1.5.

Problem 3:
For an ideal telescope, the limiting resolution is the size of the Airy disk. However, turbulence in Earth's atmosphere produces pockets of denser air which refract light rays from distant stars, causing them to strike a telescope detector at slightly different locations. At sea level, this imposes an effective resolution limit of 1 arc-second (1/3600 degree) regardless of the size of the telescope. For what diameter of telescope does this represent the diffraction limit for blue light ($\lambda =$ 400 nanometers)?

Solution:
Diffraction limited resolving power: $\theta_{min} = 1.22\ \lambda/D$
$D = 1.22\ \lambda/\theta_{min}$. Here $\theta_{min} = 4.85*10^{-6}$ rad, so the diameter of the telescope
$D = (1.22*4*10^{-7}/4.85*10^{-6})$ m = 10 cm.

Problem 4:
In the circuit shown in the figure, switch S is closed at time t = 0.
(a) Find the current reading of each meter just after S is closed.
(b) What does each meter read long after S is closed?



Solution:
Transient in RL circuits
At t = 0, the inductor acts like an open circuit, a long time after the switch is closed the inductor acts like a short circuit.
(a) A_2 an A_3 read zero, A_1 an A_4 read $I = (25\text{ V})/(55\ \Omega) = 0.45$ A.
(b) $R_{eff} = (40 + 30/11)\ \Omega = (470/11)\ \Omega$. $I_{1f} = 55/94$ A = 0.585 A.
The voltage across the 5 Ω, 10 Ω, and 15 Ω resistors in parallel is 1.6 V.
$I_2 = 0.319$ A, $I_3 = 0.16$ A, $I_4 = 0.106$ A
A_1 reads 0.585 A, A_2 reads 0.319 A, A_2 reads 0.16 A, A_2 reads 0.106 A.

Problem 5:

The Sun's spectrum, $I(h\nu)$, peaks at 1.4 eV, the spectrum of Sirius A peaks at 2.4 eV, and the luminosity (total amount of energy radiated per unit time) of Sirius A is 24 times larger than that of the Sun. Compute the diameter of Sirius A in units of the Sun's diameter.

Solution:
Stefan-Boltzmann constant: Radiated power = $\sigma * T^4 *$ Area.
Wien law: ν_{max} = constant $*T$
$T_{Sun}/T_{Sirius\ A} = 1.4/2.4 = 0.583$
$L_{Sirius\ A}/L_{Sun} = (T_{Sirius\ A}^4 D_{Sirius\ A}^2)/(T_{Sun}^4 D_{Sun}^2) = 24$
$D_{Sirius\ A}/D_{Sun} = \sqrt{(24)} * (T_{Sun}/T_{Sirius\ A})^2 = 1.67$

Problem 6:

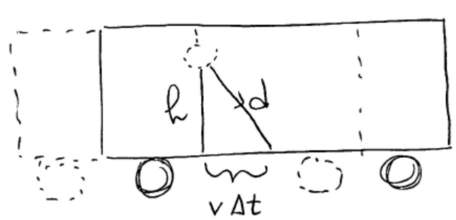
A certain oscillator satisfies the equation $d^2x/dt^2 + 4x = 0$.
Initially the particle is at the point $x = \sqrt{3}$ when it is projected towards the origin with speed $u = 2$.
(a) Find $x(t)$.
(b) How long does it take for the particle to first reach the origin?

Solution:
Harmonic motion
(a) Given: $d^2x/dt^2 + 4x = 0$, $x(0) = 3$, $(dx/dt)(0) = -2$
$x(t) = A\cos(2t + \varphi)$, $dx/dt = -2A\sin(2t + \varphi)$
Initial conditions: $\sqrt{3} = A\cos(\varphi)$, $2 = 2A\sin(\varphi)$
$\tan(\varphi) = 1/\sqrt{3}$, $\varphi = 30$ deg $= \pi/6$, $A = \sqrt{3}/\cos(\pi/6) = 2$.
$x(t) = 2\cos(2t + \pi/6)$
(b) $x(t) = 0 \dashrightarrow \cos(2t + \pi/6) = 0$, $2t + \pi/6 = \pi/6$, $t = \pi/6$ units $= 0.524$ units.

Problem 7:

Consider a light bulb inside a train moving with velocity v with respect to the train station. The bulb is switched on, and its light hits the floor of the wagon right underneath the bulb. The bulb-floor distance is h. Using only the postulate of relativity about the speed of light, calculate the time it takes for the light to hit the floor from the perspective of an observer at rest inside the train and from the perspective of an observer at rest at the station and compare those times.

Solution:
Postulate: In vacuum, light propagates with respect to any inertial frame and in all directions with the universal speed c.



From the perspective of an observer at rest inside the train:
$\Delta t_{inside} = h/c$
Prom the perspective of an observer at rest at the station:
The observer sees the light pulse move with speed c. But the light pulse is not moving a distance h, but a distance $(h^2 + v^2(\Delta t'^2))^{1/2}$.
$c\Delta t' = (h^2 + v^2(\Delta t'^2))^{1/2}$. $\Delta t' = hd/(c^2 - v^2)^{1/2} = \gamma(h/c) = \gamma\Delta t_{inside}$, with $\gamma = (1 - v^2/c^{2)-1/2}$.
We have $\Delta t = \gamma\Delta t_{inside}$. $\Delta t > \Delta t_{inside}$. Moving clocks run slow.

Problem 8:

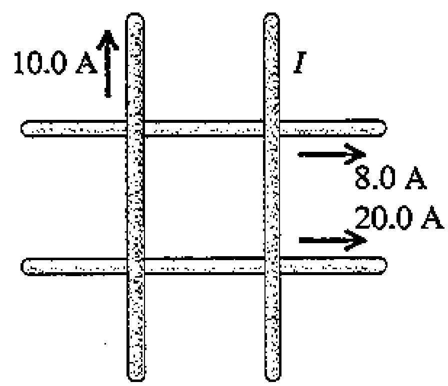
A battery with voltage V charges a capacitor with capacitance C. At $t = 0$ the battery is disconnected. The positive plate of the capacitor is then connected to one plate of a previously uncharged identical capacitor by wire with zero resistance. The negative plate of the charged capacitor is connected to the other plate of the second capacitor.

(a) What is the energy stored in the capacitors at $t = 0$?
(b) What is the energy stored in the capacitors at t --> infinity?
(c) Explain the difference from the point of view of the energy conservation.

Solution:
Capacitors in parallel, energy stored in capacitors, energy conservation
(a) For a capacitor, $U = (1/2)Q^2/C = (1/2)\, CV^2$.
$U = (1/2)Q^2/C$, $Q = CV$.
(b) $C' = 2C$ = capacitance of the capacitors in parallel. $Q' = Q$.
$U' = (1/2)Q^2/C' = (1/4)Q^2/C$, $U - U' = (1/4)Q^2/C = (1/2)U$.
(c) The electric force does work on the charges when they move. Since as t --> infinity the charges are again at rest and have no kinetic energy, the work done by the electric force is dissipated in the form of thermal energy or radiation.
eatails:
In the quasi-static approximation we have $Q_1(t)/C - IR - Q_2(t)/C = 0$.
$I = -dQ_1/dt$, $Q_1(t) + Q_2(t) = Q_0$ therefore
$dQ_1/dt = -2Q_1/(RC) + Q/(RC)$. $Q_1 = Q + (Q/2) \exp(-2t/(RC))$.
As t --> infinity, Q_1 --> Q/2, V_1 --> V/2, independent of R.
$I(t) = -(1/(RC))Q \exp(-2t/(RC))$.
Energy dissipated: $E = \int_0^\infty I^2 R dt = Q^2/(4C)$.
This is exactly the difference in the energy stored in the capacitors at $t = 0$ and at t = infinity. It is independent of R and does not change if we make R very small.

Problem 9:

Four long, current-carrying wires in the same plane intersect to form a square 40.0 cm on each side, as shown in the figure.
The magnitude and direction of the currents in three wires (arrows' directions are currents' directions) are shown in the figure.
Find the magnitude and direction of the fourth current I so that the magnetic field at the center of the square is zero.



Solution:
Ampere's law, the principle of superposition
The center of the square is the same perpendicular distance d away from each wire.
Amperes law gives the magnitude of the magnetic field produced by the current in each wire at the center of the square and the right-hand rule gives the direction. The vector sum of the fields due to the 4 wires is zero.
Let the z-axis point out of the page, the x-axis towards the right and the y-axis towards the top of the page. Then the magnetic field due to each wire has only a z-component at the center of the square, $\mathbf{B} = B_z\mathbf{k}$.
$0 = -\mu_0(10\text{ A})/(2\pi d) + \mu_0(20\text{ A})/(2\pi d) - \mu_0(8\text{ A})/(2\pi d) + \mu_0(I)/(2\pi d)$.
$I = (10 - 20 + 8)\text{ A} = -2\text{A}$
A current of 2 A flows in the negative y-direction.

Problem 10:
You have a system of two electrons whose orbital quantum numbers are $l_1 = 2$ and $l_2 = 4$ respectively.
(a) Find the possible values of l (total orbital angular momentum quantum number) for the system.
(b) Find the possible values of s (total spin angular momentum quantum number) for the system.
(c) Find the possible values of j (total angular momentum quantum number) for the system.

Solution:
Addition of angular momentum
We are supposed to add the orbital and spin angular momentum.
(a) The quantum numbers associated with the total orbital angular momentum will range from a maximum value found by adding the individual quantum numbers together to a minimum value found by taking the absolute value of the difference of the two numbers in integer steps.
$l_{max} = l_1 + l_2 = 2 + 4 = 6$
$l_{min} = | l_1 - l_2| = |2 - 4| = 2$
The possible values for L are 2, 3, 4, 5, and 6 .
(b) The quantum numbers associated with the total spin angular momentum will range from a maximum value found by adding the individual quantum numbers together to a minimum value found by taking the absolute value of the difference of the two numbers in integer steps.
$s_{max} = s_1 + s_2 = 1/2 + 1/2 = 1$
$s_{min} = |s_1 - s_2| = |1/2 - 1/2| = 0$
The possible values for S are 0 and 1.
(c) The largest possible value for j will be found by adding together the maximum values for both l and s. The minimum value for j will be found by subtracting the largest possible value of s from the smallest possible value for l.
$j_{smax} = l_{max} + s_{max} = 6 + 1 = 7$
$j_{min} = | l_{min} - s_{max}| = |2 - 1| = 1$
j = 1, 2, 3, 4, 5, 6, 7

Section II:
Work 3 out of the 5 problems, problem 11 – problem 15! (12 points each)

Problem 11:
Show that the square of the 4-vector momentum of a particle is Lorenz invariant.

Solution:
The "dot product" of two 4-vectors, $(a_0, \mathbf{a}) \cdot (b_0, \mathbf{b}) = a_0 b_0 - a \cdot \mathbf{b}$
$p^\mu = (E/c, \mathbf{p}) = (\gamma mc, \gamma m\mathbf{v})$
$(\gamma mc, \gamma m\mathbf{v}) \cdot (\gamma mc, \gamma m\mathbf{v}) = \gamma^2 m^2 c^2 - \gamma^2 m^2 v^2 = \gamma^2 m^2 (c^2 - v^2) = m^2 c^2$, since $\gamma 2 = c2/(c^2 - v^2)$.

Problem 12:
In a medium with conductivity σ but no net charge, write down Maxwell's equations, and derive the wave equation for the electric field, **E**, in this medium.

Solution:
Maxwell's equations
In regions with $\rho_f = 0$ and $\mathbf{j_f} = \sigma_c \mathbf{E}$ Maxwell's equations can be used to show that both **E** and **B**
Maxwell's equations:
$$\vec{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \vec{\nabla} \cdot \boldsymbol{B} = 0, \quad \vec{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}.$$
Assume $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\rho_f = 0$ and $\mathbf{j_f} = \sigma_c \mathbf{E}$ in the conductor. Then
$\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.
$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E} = -\partial(\boldsymbol{\nabla} \times \mathbf{B})/\partial t = -\mu_0 \partial \mathbf{j}/\partial t - \mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2$.
$\boldsymbol{\nabla}^2 \mathbf{E} = \mu_0 \sigma_c \partial \mathbf{E}/\partial t + \mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2$, or
$\boldsymbol{\nabla}^2 \mathbf{E} - \mu_0 \sigma_c \partial \mathbf{E}/\partial t - \mu_0 \varepsilon_0 \partial^2 \mathbf{E}/\partial t^2 = 0$.
This is the damped wave equation for **E**.

Problem 13:
Light from Sirius A shows a shift in wavelength due to the influence of a companion star, Sirius B, with a period of 50 years.
(a) If the Balmer α line of hydrogen (λ_{rest} = 656 nanometers) exhibits a maximum Doppler shift of 0.025 nm, what is the orbital velocity of Sirius A?
(b) Given this orbital velocity, what is the radius of Sirius's orbit, if one assumes a circular orbit?
(c) What is the combined mass of Sirius A and Sirius B?

Solution:
Doppler shift, orbiting
(a) Assume a blue shift star is approaching Earth, f' > f. $f'/f = [(1 + v/c)/(1 - v/c)]^{1/2}$.
If v << c and Δf << f, then $|\Delta f/f| = |\Delta\lambda/\lambda| = v/c$.
Here v/c = 0.025/666 = 3.81*10-5, v = 11433 m/s.
(b) $v = 2\pi r/T$, $r = vT/(2\pi) = 2.87*10^{12}$ m.
(c) Solve for the relative motion of two interacting masses m_1 and m_2 by solving for the motion of one fictitious particle of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ in a central field.
$F = Gm_1m_2/r^2 = (m_1m_2/(m_1 + m_2))v^2/r$, $m_1 + m_2 = v^2 r/G = 5.62*10^{30}$ kg

Problem 14:

A boat with mass m is slowed by a drag force F(v). Its velocity decreases according to the formula $v(t)= c^2(t - t_f)^2$ for $t \leq t_f$, where c is a constant and t_f is the time a which it stops. Find the force F(v) as function of v.

Solution:

Newton's second law

$a(t) = dv/dt = 2c^2(t - t_f)$

$F(t) = = ma(t) = 2mc^2(t - t_f)$

$t - t_f = -\sqrt{(v)}/c$ (Select the negative solution since $t - t^f$ is negative.)

$F(v) = -2mc^2\sqrt{(v)}/c = -2mc\sqrt{(v)}$

Problem 15:

In one-dimensional potential energy function of a particle as a function of x is a delta function at x = 0.

$U(x) = C\delta(x)$.

A stream of non-relativistic particles of mass m and energy E approaches the origin from one side. Derive an expression for the transmission probability T(E). Does T(E) depend on the sign of C?

Solution:

This is a "square potential" problem. We solve $H\Phi(x) = E\Phi(x)$ in regions where V(x) is constant and apply boundary conditions. V(x) = 0 everywhere except at x = 0.

$\Phi_1(x) = A_1 \exp(ikx) + A_1'\exp(-ikx)$ for x < 0. $k^2 = 2mE/\hbar^2$.

$\Phi_2(x) = A_2 \exp(ikx)$ for x > 0.

Φ is continuous at x = 0. $\Phi_1(0) = \Phi_2(0)$. $A_1 + A_1' = A_2$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{2m}{\hbar^2}\left(E - V(x)\right)\phi = 0 .$$

Let us evaluate this equation at x = ε and at x = -ε and write down a difference equation.

$$\left[\frac{\partial \phi_2(\varepsilon)}{\partial x} - \frac{\partial \phi_1(-\varepsilon)}{\partial x}\right]_{\varepsilon\to 0} = \left[-\frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}\left(E - C\delta(x)\right)\phi(x)dx\right]_{\varepsilon\to 0} = \frac{2mC}{\hbar^2}\phi(0)$$

If V does not remain finite at the step, then ∂Φ/∂x has a finite discontinuity at the step.

$iA_1k - iA_1'k = iA_2k - A_2 2mC/\hbar^2$, $A_1 - A_1' = A_2 - A_2 2mC/(ik\hbar^2) = [1 - 2mC/(ik\hbar^2)]A_2$

Eliminate A_1':

$2A_1 = [2 - 2mC/(ik\hbar^2)]A_2$, $A_2/A_1 = 1/[1 - mC/(ik\hbar^2)] = ik\hbar^2/(ik\hbar^2 - mC)$

$T(E) = (k|A_1|^2)/(k|A_2|^2) = \hbar^4k^2/(\hbar^4k^2 + m^2C^4) = E/(E + mC^2/(2\hbar^2))$

T(E) does not depend on the sign of C.