Homework 2, solutions

problem 1, solution:

Let $x < 0$ denote region 1 and $x > 0$ denote region 2.

Let $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$.

Then $\phi_1(x) = A_1 e^{ik_1x} + A_1' e^{-ik_1x}$, $\phi_2(x) = A_2 e^{ik_2x}$.

ϕ and $\frac{\partial\phi}{\partial x}$ are continuous at $x = 0$. This implies

$$\frac{A_1'}{A_1} = \frac{k_1 - k_2}{k_1 + k_2} , \quad \frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2} .$$

The probability of reflection is $R = \left|\frac{A_1'}{A_1}\right|^2 = 1 - \frac{4k_1k_2}{(k_1+k_2)^2}$.

$$k_1 = \sqrt{\frac{8mV_0}{\hbar^2}} , \quad k_2 = \sqrt{\frac{6mV_0}{\hbar^2}} .$$

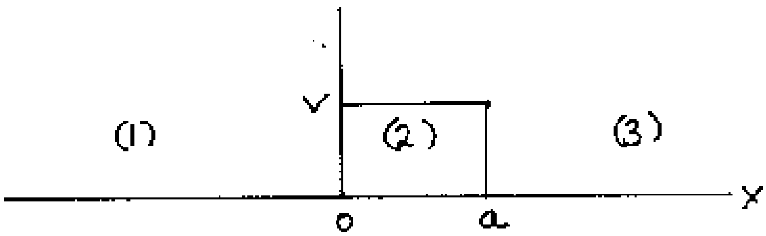
$$k_1 k_2 = \sqrt{48}\,\frac{mV_0}{\hbar^2} . \qquad k_1 + k_2 = \left(\sqrt{8} + \sqrt{6}\right)\sqrt{\frac{mV_0}{\hbar^2}} .$$

$$R = 1 - \frac{\sqrt{48}\cdot 4}{(\sqrt{8}+\sqrt{6})^2} = .005 .$$

problem 2, solution:

This is a "square potential" problem.

a)



$E < V$. Let $k = \sqrt{\frac{2mE}{\hbar^2}}$, $\rho = \sqrt{\frac{2m(V-E)}{\hbar^2}}$. Then

$\phi_1 = A_1 e^{ikx} + A_1' e^{-ikx}$, $\phi_2 = B_2 e^{\rho x} + B_2' e^{-\rho x}$, $\phi_3 = A_3 e^{ikx}$, if no particles are incident from $-\infty$. The boundary conditions are that $\phi(x)$ and $\frac{\partial \phi}{\partial x}$ are continuous at $x = 0$ and $x = a$.

We want to find A_3 and A_1' in terms of A_1.

at $x = a$: $B_2 e^{\rho a} + B_2' e^{-\rho a} = A_3 e^{ika}$, $\rho B_2 e^{\rho a} - \rho B_2' e^{-\rho a} = ik A_3 e^{ika}$.

Solve for B_2 and B_2' in terms of A_3.

$B_2 e^{\rho a} + B_2 e^{\rho a} - \frac{ik}{\rho} A_3 e^{ika} = A_3 e^{ika}$, $B_2' e^{-\rho a} + \frac{ik}{\rho} A_3 e^{ika} + B_2' e^{-\rho a} = A_3 e^{ika}$,

$B_2 = \frac{1}{2}\left(\frac{\rho + ik}{\rho}\right) e^{ika - \rho a} A_3 = C_2 A_3$, $B_2' = \frac{1}{2}\left(\frac{\rho - ik}{\rho}\right) e^{ika + \rho a} A_3 = C_2' A_3$.

at $x = 0$: $A_1 + A_1' = B_2 + B_2' = (C_2 + C_2')A_3$, $ik(A_1 - A_1') = \rho(B_2 - B_2') = \rho(C_2 - C_2')A_3$.

Solve for A_1' and A_3 in terms of A_1.

$2A_1 = \left[C_2 + C_2' + \frac{\rho}{ik}(C_2 - C_2')\right] A_3$, $2A_1' = \left[C_2 + C_2' - \frac{\rho}{ik}(C_2 - C_2')\right] A_3$

$$= \frac{\left[C_2 + C_2' - \frac{\rho}{ik}(C_2 - C_2')\right]}{\left[C_2 + C_2' + \frac{\rho}{ik}(C_2 - C_2')\right]} 2A_1 .$$

$$C_2+C_2' = \frac{e^{ika}}{2\rho}\left[(\rho+ik)e^{-\rho a}+(\rho-ik)e^{\rho a}\right] = \frac{e^{ika}}{2\rho}\left[\rho(e^{-\rho a}+e^{+\rho a})+ik(e^{-\rho a}-e^{+\rho a})\right]$$

$$= \frac{e^{ika}}{\rho}\left[\rho\cosh\rho a - ik\sinh\rho a\right].$$

$$C_2-C_2' = \frac{e^{ika}}{\rho}\left[-\rho\sinh\rho a + ik\cosh\rho a\right].$$

$$2A_1 = \frac{e^{ika}}{\rho}\left[\rho\cosh\rho a - ik\sinh\rho a - \frac{\rho^2}{ik}\sinh\rho a + \rho\cosh\rho a\right]A_3 .$$

$$A_1 = e^{ika}\left[\cosh\rho a - i\left(\frac{k^2-\rho^2}{2k\rho}\right)\sinh\rho a\right]A_3 . \qquad (\cosh^2 x - \sinh^2 x = 1).$$

$$T = \frac{k}{k}\left|\frac{A_3}{A_1}\right|^2, \quad |A_1|^2 = \left[\cosh^2\rho a + \left(\frac{k^2-\rho^2}{2k\rho}\right)^2\sinh^2\rho a\right]|A_3|^2$$

$$= \left[1+\left[\left(\frac{k^2-\rho^2}{2k\rho}\right)^2+\frac{4k^2\rho^2}{4k^2\rho^2}\right]\sinh^2\rho a\right]|A_3|^2 = \left[1+\frac{(k^2+\rho^2)^2}{4k^2\rho^2}\sin^2\rho a\right]|A_3|^2 .$$

$$T = \frac{4k^2\rho^2}{4k^2\rho^2+(k^2+\rho^2)^2\sinh^2\rho a} = \frac{4E(V-E)}{4E(V-E)+V^2\sinh^2\left(\sqrt{\frac{2m(V-E)}{\hbar^2}}\,a\right)} .$$

$$C_2+C_2' - \frac{\rho}{ik}(C_2-C_2') = \frac{e^{ika}}{\rho}\,\frac{(\rho^2+k^2)}{ik}\sinh\rho a .$$

$$\frac{C_2+C_2'+\frac{\rho}{ik}(C_2-C_2')}{C_2+C_2'-\frac{\rho}{ik}(C_2-C_2')} = \frac{2\rho\cosh\rho a + \frac{(k^2-\rho^2)}{ik}\sinh\rho a}{\frac{(k^2+\rho^2)}{ik}\sinh\rho a} = \frac{2ik\rho\cosh\rho a+(k^2-\rho^2)\sinh\rho a}{(k^2+\rho^2)\sinh\rho a} .$$

$$|A_1'|^2 = \frac{(k^2+\rho^2)^2\sinh^2\rho a\,|A_1|^2}{4k^2\rho^2\cosh^2\rho a+(k^2-\rho^2)^2\sinh^2\rho a} = \frac{(k^2+\rho^2)^2\sinh^2\rho a\,|A_1|^2}{4k^2\rho^2+(k^2+\rho^2)^2\sinh^2\rho a} .$$

$$R = \frac{k}{k}\left|\frac{A_1'}{A_1}\right|^2 = \frac{V^2\sinh^2\left(\sqrt{\frac{2m(V-E)}{\hbar^2}}\,a\right)}{4E(E-V)+V^2\sinh^2\left(\sqrt{\frac{2m(V-E)}{\hbar^2}}\,a\right)} .$$

b) $$T+R = \frac{4k^2\rho^2}{4k^2\rho^2+(k^2+\rho^2)^2\sinh^2\rho a} + \frac{(k^2+\rho^2)^2\sinh^2\rho a}{4k^2\rho^2+(k^2+\rho^2)^2\sinh^2\rho a} = 1 .$$

problem 3, solution:

Let region 1 extend from $x=0$ to $x=a$ and region 2 from $x=a$ to $x=\infty$.

a) $E<V_1$. Define $k=\sqrt{\frac{2m}{\hbar^2}(E+V_1)}$, $\rho=\sqrt{\frac{2m}{\hbar^2}(-E)}$, $k_0=\sqrt{\frac{2mV_1}{\hbar^2}}$.

$\phi_1 = A_1 e^{ikx} + A_1' e^{-ikx}$, $\phi_1(0)=0$, $A_1 + A_1' = 0$, $A_1 = -A'$.

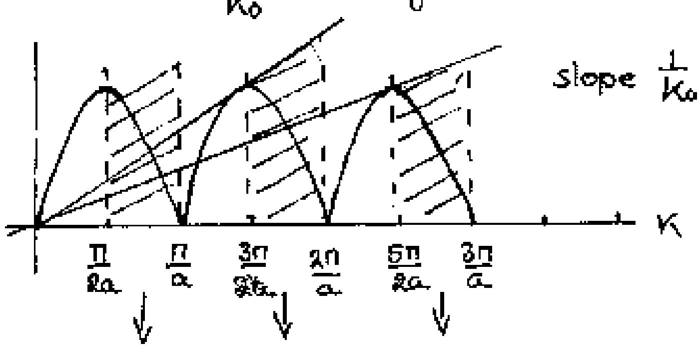
$\phi_1 = A_1(e^{ikx} - e^{-ikx}) = 2iA_1 \sin kx = A \sin kx$. $A = 2iA_1$.

$\phi_2 = \cancel{B_2 e^{\rho x}} + B_2' e^{-\rho x}$. $\phi_1(a) = \phi_2(a)$. $\left.\frac{\partial \phi_1}{\partial x}\right|_a = \left.\frac{\partial \phi_2}{\partial x}\right|_a$.

$$\left.\begin{aligned} A\sin ka &= B_2' e^{-\rho a} \\ A k\cos ka &= -\rho B_2' e^{-\rho a} \end{aligned}\right\}$$ A solution only exists if $\sin ka = -\frac{k}{\rho}\cos ka$ or $\cot ka = -\frac{\rho}{k}$.

$$\frac{1}{\sin^2(ka)} = 1+\cot^2(ka) = \frac{k^2+\rho^2}{k^2} = \frac{k_0^2}{k^2}.$$

$|\sin ka| = \frac{k}{k_0}$ in regions where $\cot ka < 0$.



slope $\frac{1}{k_0}$.

For two and only two bound states to exist we need

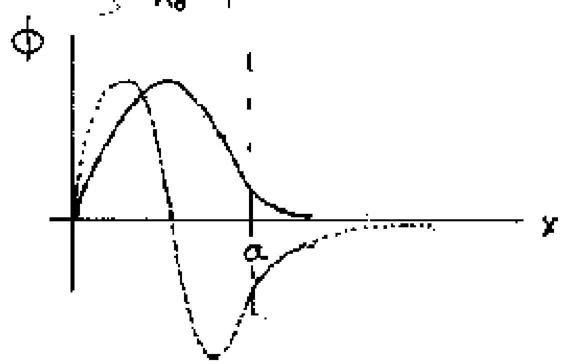
$$\frac{2a}{5\pi} < \frac{1}{k_0} < \frac{2a}{3\pi}.$$

$$\frac{3}{2}\frac{\pi}{a} < k_0 < \frac{5}{2}\frac{\pi}{a},$$

$$\frac{3}{2}\frac{\pi}{a} < \sqrt{\frac{2mV_1}{\hbar^2}} < \frac{5}{2}\frac{\pi}{a}.$$

In these regions $\cot ka < 0$.

b) Inside the well $\phi_1 = A\sin kx$, outside the well $\phi_2 = A\sin ka\, e^{\rho a} e^{-\rho x}$, where k is a solution of $|\sin ka| = \frac{k}{k_0}$, $\cot ka < 0$, and ρ is determined once k is determined.



problem 4, solution

a) ϕ is continuous at $x=0$. (region 1: $x<0$; region 2: $x>0$)

$\frac{\partial\phi}{\partial x}$ has a finite discontinuity at $x=0$.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{2m}{\hbar^2}[E-V(x)]\phi=0 . \quad \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)=-\frac{2m}{\hbar^2}[E-V(x)]\phi(x).$$

$$\frac{\partial\phi_2(\varepsilon)}{\partial x}-\frac{\partial\phi_1(-\varepsilon)}{\partial x}=-\frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}(E+aV_0\delta(x))\phi(x)dx \underset{\varepsilon\to 0}{=} -\frac{2m}{\hbar^2}aV_0\phi(0) .$$

For $x\neq 0$: region 1: $\phi_1(x)=A_1e^{ikx}+A_1'e^{-ikx}$, $k=\sqrt{\frac{2mE}{\hbar^2}}$.

region 2: $\phi_2(x)=A_2e^{ikx}$.

b) $T=\frac{k}{k}\frac{|A_2|^2}{|A_1|^2}$. $R=\frac{k}{k}\frac{|A_1'|^2}{|A_1|^2}$. To find A_2 and A_1' in terms of A_1 we apply the boundary conditions.

$$A_1+A_1'=A_2, \quad iA_1k-iA_1'k=iA_2k+\frac{2m}{\hbar^2}aV_0A_2 .$$

$$A_1-A_1'=\left(1+\frac{2maV_0}{ik\hbar^2}\right)A_2, \quad \frac{A_2}{A_1}=\frac{1}{1+\frac{maV_0}{\hbar^2ki}}=\frac{i\hbar^2k}{maV_0+i\hbar^2k} .$$

$$T=\frac{\hbar^4k^2}{\hbar^4k^2+m^2a^2V_0^2}=\frac{\hbar^4\frac{2mE}{\hbar^2}}{\hbar^4\frac{2mE}{\hbar^2}+m^2a^2V_0^2}=\frac{E}{E+\frac{ma^2V_0^2}{2\hbar^2}} .$$

$$R=1-T=\frac{ma^2V_0^2}{2\hbar^2E+ma^2V_0^2} .$$

c) As $E\to 0$ $R\to 1$; $T\to 0$.
As $E\to\infty$ $T\to 1$; $R\to 0$.

d) As $E\to 0$ nothing is transmitted past the first δ-function.
As $E\to\infty$ everything is transmitted across both δ-functions.

problem 5, solution

a) $\lambda = \frac{h}{p} = \frac{h}{\gamma m v} \propto \frac{1}{\gamma v} = \frac{\sqrt{1-v^2/c^2}}{v} = \sqrt{\frac{1}{v^2}-\frac{1}{c^2}}$.

b) nonrelativistic: $T = 15eV = \frac{p^2}{2m_p}$ $\quad \Lambda = \frac{h}{\sqrt{2m_p 15eV}} = \frac{hc}{\sqrt{2m_p c^2 15eV}} = \frac{1.24\times10^4 eV\text{-}\mathring{A}}{\sqrt{30\times938.28\times10^6 eV}}$

$= 7.4\times10^{-2}\mathring{A}$

c) nonrelativistic: $\lambda = \frac{h}{\sqrt{2m_e 15keV}} = \frac{1.24\times10^4 eV\mathring{A}}{\sqrt{15\times10^3 \cdot 2\times.511\times10^6 ev^2}} = .1\mathring{A}$

relativistic: $E = m_o c^2 + T \qquad E^2 = m_o^2c^4 + T^2 + 2m_oc^2T$

$E^2 = p^2c^2 + m_o^2c^4$

$p^2c^2 = T^2 + 2m_oc^2T$

$\lambda = \frac{h}{\sqrt{T^2/c^2 + 2m_eT}} = \frac{1.24\times10^4\ eV\text{-}\mathring{A}}{\sqrt{(15\times10^3)^2 + 2\cdot.511\times10^6\times15\times10^3}} = .1\mathring{A}$

d) relativistic: $\lambda = \frac{h}{\gamma m_e v} = \frac{hc^2\sqrt{1-v^2/c^2}}{m_e c^2 v} = \frac{1.24\times10^4 ev\text{-}\mathring{A}\ \sqrt{1-9\times10^{-2}}}{.511\times10^6 eV} \frac{3\times10^8 m/s}{9\times10^7 m/s} = 7.7\times10^{-2}\mathring{A}$

e) nonrelativistic: $\lambda = \frac{h}{mv} = \frac{6.626\times10^{-34} Js}{1kg\ 1m/s} = 6.626\times10^{-34} m$

f) nonrelativistic: $\lambda = \frac{h}{\sqrt{2m_n \frac{1}{25}eV}} = \frac{1.24\times10^4 eV\mathring{A}}{\sqrt{\frac{2}{25}\ 939.57\times10^6 ev^2}} = 1.4\mathring{A}$

g) $T = eV$ $\qquad$ nonrelativistic: $\Lambda = \frac{h}{\sqrt{2m_e T(ev)}} = \frac{1.24\times10^4 eV\text{-}\mathring{A}}{\sqrt{2m_ec^2}\sqrt{T}} = 12.27\frac{1}{\sqrt{T(ev)}}\mathring{A}$

relativistic: $\Lambda = \frac{h}{\sqrt{T^2/c^2 + 2m_eT}} = \frac{hc\ (eV\text{-}\mathring{A})}{\sqrt{\underbrace{T^2}_{ev^2} + \underbrace{2m_ec^2}_{ev}\underbrace{T}_{eV}}}$

h) $\Lambda = \frac{hc}{\sqrt{T^2 + 2mc^2T}}$