Homework 4, solutions

problem 1, solution

a) $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = |A|^2 \int_{-\infty}^{+\infty} dx\, e^{-x^2/b^2} = |A|^2\, b\sqrt{\pi}$. $\quad |A|^2 = \frac{1}{b\sqrt{\pi}}$.

Possible A: $A = \frac{1}{\sqrt{b}\,\pi^{1/4}}$.

b) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$. $\langle x\rangle = 0$ from symmetry.

$\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2 |\psi(x)|^2 dx = \frac{1}{b\sqrt{\pi}} \int_{-\infty}^{+\infty} dx\, x^2 e^{-x^2/b^2} = \frac{b^3}{b\sqrt{\pi}} \underbrace{\int_{-\infty} dx'\, x'^2 e^{-x'^2}}_{\sqrt{\pi}/2} = \frac{b^2}{2}$.

$\Delta x = \frac{b}{\sqrt{2}}$.

c) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$.

$p\,\psi(x) = \frac{\hbar}{i}\frac{d}{dx}\psi(x) = \left(p_0 + i\hbar\frac{x}{b^2}\right)\psi(x)$.

$p^2\psi(x) = \left[\left(p_0 + i\hbar\frac{x}{b^2}\right)^2 + \frac{\hbar^2}{b^2}\right]\psi(x)$.

Therefore

$\langle p\rangle = \int_{-\infty}^{+\infty} dx\, \left(p_0 + i\hbar\frac{x}{b^2}\right)|\psi(x,0)|^2 = p_0$,

$\langle p^2\rangle = \int_{-\infty}^{+\infty} dx\, \left[(p_0 + i\hbar x/b^2)^2 + \frac{\hbar^2}{b^2}\right]|\psi(x,0)|^2 = p_0^2 + \frac{\hbar^2}{b^2} - \frac{\hbar^2}{b^4}\langle x^2\rangle = p_0^2 + \frac{\hbar^2}{b^2} - \frac{\hbar^2}{2b^2}$

$= p_0^2 + \hbar^2/2b^2$.

$\Delta p = \frac{\hbar}{\sqrt{2}\,b}$. $\quad \Delta x\,\Delta p = \frac{b}{\sqrt{2}}\frac{\hbar}{\sqrt{2}\,b} = \frac{\hbar}{2}$.

d) $\frac{\Delta v}{c} = \frac{\Delta p}{m_e c} = \frac{\hbar}{2 m_e c\,\Delta x} = \frac{1.05\times 10^{-34}}{2\times 9.1\times 10^{-31}\times 3\times 10^{8}\times 10^{-10}} = 1.9\times 10^{-3}$.

problem 2, solution

For the infinite square well $\psi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$.

a) $\langle H \rangle = \frac{1}{2}(E_1+E_2) = \frac{1}{2}\frac{\pi^2\hbar^2}{2mL^2}(1^2+2^2) = \frac{5}{4}\frac{\pi^2\hbar^2}{mL^2}$.

b) $\psi(x,t) = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\left[\sin\frac{\pi x}{L} e^{-iE_1t/\hbar} + \sin\frac{2\pi x}{L} e^{-iE_2t/\hbar}\right]$ $\quad E_1 = \frac{\pi^2\hbar^2}{2mL^2}$, $E_2 = \frac{4\pi^2\hbar^2}{2mL^2}$.

We could insert $e^{i\phi}$, ϕ = arbitrary phase factor,
or multiply the whole expression by $e^{i\chi}$, χ = arbitrary phase factor.

c) $\langle p_x \rangle = \int \Psi^*(x,t)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t)\,dx$

$$= \frac{1}{L}\frac{\hbar}{i}\int_0^L \left[\sin\frac{\pi x}{L} e^{+iE_1t/\hbar} + \sin\frac{2\pi x}{L} e^{+iE_2t/\hbar}\right]\frac{\partial}{\partial x}\left[\sin\frac{\pi x}{L} e^{-iE_1t/\hbar} + \sin\frac{2\pi x}{L} e^{-iE_2t/\hbar}\right]dx$$

$$= \frac{-i\hbar}{L}\left[\int_0^L \sin\frac{2\pi x}{L}\cos\frac{\pi x}{L}\frac{\pi}{L} e^{i(E_2-E_1)t/\hbar}\,dx + \int_0^L \sin\frac{\pi x}{L}\cos\frac{2\pi x}{L}\frac{2\pi}{L} e^{-i(E_2-E_1)t/\hbar}\,dx\right]$$

$$= \frac{-i\hbar\pi}{L^2} e^{i(E_2-E_1)t/\hbar}\frac{L}{\pi}\underbrace{\int_0^\pi \cos x' \sin 2x'\,dx'}_{\frac{4}{3}} - \frac{i\hbar 2\pi}{L^2} e^{-i(E_2-E_1)t/\hbar}\underbrace{\int_0^\pi \cos 2x' \sin x'\,dx'}_{-\frac{2}{3}}$$

$$= \frac{-i\hbar}{L}\frac{4}{3}\left(e^{i(E_2-E_1)t/\hbar} - e^{-i(E_2-E_1)t/\hbar}\right) = \frac{8\hbar}{3L}\sin\frac{(E_2-E_1)t}{\hbar}.$$

problem 3 solution

a) H is a Hermitian operator, its eigenvalues are real.

$\langle \psi_n | H | \psi_m \rangle = E_m \langle \psi_n | \psi_m \rangle$

$\langle \psi_n | H^{\dagger} | \psi_m \rangle = E_n \langle \psi_n | \psi_m \rangle$ Assume $E_m \neq E_n$.

But since $H = H^{\dagger}$. $(E_m - E_n)\langle \psi_n | \psi_m \rangle = 0 \Rightarrow \langle \psi_n | \psi_m \rangle = 0$.

b) Yes. An orthonormal basis for the state space $\mathcal{E}$ can be formed by the eigenfunctions of any Hermitian operator.

c) Any wave function can be expanded in terms of eigenfunctions of H.

$|\psi_{trial}\rangle = \sum_n a_n |\psi_n\rangle$, $\{|\psi_n\rangle\}$ orthonormal basis for the state space $\mathcal{E}$.

$H|\psi_{trial}\rangle = \sum_n a_n E_n |\psi_n\rangle$.

$$\langle \psi_{trial} | H | \psi_{trial} \rangle = \sum_n a_n E_n \langle \psi_{trial} | \psi_n \rangle = \sum_n \sum_m a_n a_m^* E_n \underbrace{\langle \psi_m | \psi_n \rangle}_{\delta_{mn} \langle \psi_n | \psi_n \rangle}$$

$$= \sum_n |a_n|^2 E_n \langle \psi_n | \psi_n \rangle \geq E_0 \sum_n |a_n|^2 \langle \psi_n | \psi_n \rangle,$$

since $E_0 \leq E_n$.

$$\langle \psi_{trial} | H | \psi_{trial} \rangle \geq E_0 \langle \psi_{trial} | \psi_{trial} \rangle.$$

problem 4, solution

a) $H|E\rangle = E|E\rangle$; $|E\rangle$ is an eigenvector with eigenvalue E.

$\langle E|H|E\rangle = \langle E|HE\rangle = E\langle E|E\rangle$

$= \underbrace{\langle HE|E\rangle}_{\text{since } H = H^{\dagger}} = E^{*}\langle E|E\rangle$; $(E^{*}-E)\langle E|E\rangle = 0$, $\langle E|E\rangle \neq 0 \Rightarrow E = E^{*}$.

b) $H|E'\rangle = E'|E'\rangle$; $|E'\rangle$ is an eigenvector with eigenvalue E'.

$\langle E'|H|E\rangle = \langle E'|HE\rangle = E\langle E'|E\rangle$

$= \langle HE'|E\rangle = E'\langle E'|E\rangle$; $(E'-E)\langle E'|E\rangle = 0$.

If $E' \neq E$ then $\langle E'|E\rangle = 0$.

c) $[H, L^2] = HL^2 - L^2H$.

$HL^2|E\ell m\rangle = \hbar^2\ell(\ell+1)$, $\quad L^2H = \hbar^2\ell(\ell+1)E|E\ell m\rangle$.

$L^2H|E\ell m\rangle = HL^2|E\ell m\rangle = E\hbar^2\ell(\ell+1)|E\ell m\rangle$.

$[H, L^2]|E\ell m\rangle = 0$.

Consider an arbitrary vector $|\psi\rangle$. $|\psi\rangle = \sum c_{E\ell m}|E\ell m\rangle$ since $\{|E\ell m\rangle\}$ is a complete set of states, i.e. a basis for the state space.

$[H, L^2]|\psi\rangle = \sum c_{E\ell m}\underbrace{[H, L^2]|E\ell m\rangle}_{0} = 0$ for any $|\psi\rangle$.

Therefore $[H, L^2] = 0$.

Similarly, $HL_z|E\ell m\rangle = L_zH|E\ell m\rangle = m\hbar|E\ell m\rangle$.

$[H, L_z]|E\ell m\rangle = 0$, $\quad [H, L_z]|\psi\rangle = \sum c_{E\ell m}\underbrace{[H, L_z]|E\ell m\rangle}_{0} = 0$

for any $|\psi\rangle$. Therefore $[H, L_z] = 0$.