Homework 5, solutions

problem 1, solution

Let $\{|a_1\rangle, |a_2\rangle\}$ be an orthonormal eigenbasis of A, $A|a_i\rangle = a_i|a_i\rangle$ and $\{|b_1\rangle, |b_2\rangle\}$ be an orthonormal eigenbasis of B, $B|b_i\rangle = b_i|b_i\rangle$.

$|b_1\rangle = \cos\theta\,|a_1\rangle + \sin\theta\, e^{i\phi}|a_2\rangle$, $\langle b_1|b_2\rangle = 0$, $\langle b_1|b_1\rangle = \langle b_2|b_2\rangle = 1$,
$|b_2\rangle = -\sin\theta\,|a_1\rangle + \cos\theta\, e^{i\phi}|a_2\rangle$,
is the most general expansion of the $|b_i\rangle$ in terms of the $|a_i\rangle$.

The initial state can be written as some linear combination of $|a_1\rangle$ and $|a_2\rangle$.
$|\psi\rangle = \Lambda_1|a_1\rangle + \Lambda_2|a_2\rangle$ $\quad |\Lambda_1|^2 + |\Lambda_2|^2 = 1$

Branch 1:
Assume the first measurement yields a_1. The probability of this is $|\Lambda_1|^2$.
What is the probability of measuring a_1 again after measuring B?

$P_{a_1}(b_1) = \cos^2\theta$, (system is now in $|b_1\rangle$), $P_{b_1}(a_1) = \cos^2\theta$, $P_{a_1}(b_1)P_{b_1}(a_1) = \cos^4\theta$.

$P_{a_1}(b_2) = \sin^2\theta$, (system is now in $|b_2\rangle$), $P_{b_2}(a_1) = \sin^2\theta$, $P_{a_1}(b_2)P_{b_2}(a_1) = \sin^4\theta$.

Probability of first path + probability of second path $= \cos^4\theta + \sin^4\theta$.
Total probability of this branch $= |\Lambda_1|^2(\cos^4\theta + \sin^4\theta)$

Branch 2:
Assume the first measurement yields a_2. The probability of this is $|\Lambda_2|^2$.
What is the probability of measuring a_2 again after measuring B?

$P_{a_2}(b_1) = \sin^2\theta$, (system is now in $|b_1\rangle$), $P_{b_1}(a_2) = \sin^2\theta$ $\quad P_{a_2}(b_1)P_{b_1}(a_2) = \sin^4\theta$

$P_{a_2}(b_2) = \cos^2\theta$, (system is now in $|b_2\rangle$), $P_{b_2}(a_2) = \cos^2\theta$ $\quad P_{a_2}(b_2)P_{b_2}(a_2) = \cos^4\theta$

Probability of first path + probability of second path $= \sin^4\theta + \cos^4\theta$
Total probability of this branch $= |\Lambda_2|^2(\cos^4\theta + \sin^4\theta)$

Probability of obtaining same result $= \left(|\Lambda_1|^2 + |\Lambda_2|^2\right)(\cos^4\theta + \sin^4\theta) = \cos^4\theta + \sin^4\theta$, independent of initial state.

problem 2, solution

The eigenfunctions of H for the infinite square well ($V(x) = 0 \quad 0<x<a$; $V(x) = \infty$ everywhere else) are:

$$\phi_n(x) = \sqrt{\frac{2}{a}}\,\sin\frac{n\pi x}{a} \quad , \quad \text{with eigenvalues } \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

$$\langle x\rangle_n = \langle\phi_n|x|\phi_n\rangle = \frac{2}{a}\int_0^a \sin^2\frac{n\pi x}{a}\,x\,dx = \frac{a}{2}.$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \qquad \langle x^2\rangle_n = \langle\phi_n|x^2|\phi_n\rangle = \frac{2}{a}\int_0^a \sin^2\frac{n\pi x}{a}\,x^2\,dx.$$

$$\langle x^2\rangle_n = \frac{2}{a}\left(\frac{x^3}{6} - \left(\frac{x^2a}{4n\pi} - \frac{a^2}{8n^2\pi^2}\right)\sin\frac{2n\pi x}{a} - \frac{a^2x\cos\frac{2n\pi x}{a}}{4n^2\pi^2}\right)\Bigg|_0^a$$

$$= \frac{2}{a}\left(\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\right) = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}.$$

$$(\Delta x)^2 = \frac{a^2}{3} - \frac{a^2}{4} - \frac{a^2}{2n^2\pi^2} \xrightarrow[n\to\infty]{} \frac{a^2}{12}.$$

Classically the particle bounces back and forth. It spends the same amout of time at each position.

Therefore $\bar{x} = \frac{a}{2}$ for a large # of particles with arbitrary initial positions.

$$x_{rms}^2 = \frac{1}{a}\int_0^a \left(x - \frac{a}{2}\right)^2 dx = \frac{x^3}{3} - \frac{ax^2}{2} + \frac{a^2x}{4}\Bigg|_0^a = \frac{a^2}{12}.$$

problem 3, solution

Initially the system is in state $|\psi_1\rangle$. $H_0|\psi_1\rangle = E_1|\psi_1\rangle$,

$H_0|\psi_2\rangle = E_2|\psi_2\rangle$.

$$P(E_2,t) = \frac{1}{\hbar^2}\left|\int_0^t e^{i(E_2-E_1)t'/\hbar}\langle\psi_2|W(t')|\psi_1\rangle dt'\right|^2.$$

$W(t') = \varepsilon V_0 \cos\omega t'$.

$\langle\psi_2|W(t')|\psi_1\rangle = \varepsilon\cos\omega t'\langle\psi_2|V_0|\psi_1\rangle$.

$$\int_0^t e^{i(E_2-E_1)t'/\hbar}\langle\psi_2|W(t')|\psi_1\rangle dt' = \left[\int_0^t e^{i\omega_{21}t'}\cos\omega t'\,dt'\right]\varepsilon\langle\psi_2|V_0|\psi_1\rangle.$$

$$\int_0^t e^{i\omega_{21}t'}\cos\omega t'\,dt' = \frac{1}{2}\int_0^t e^{i(\omega_{21}+\omega)t'}dt' + \frac{1}{2}\int_0^t e^{i(\omega_{21}-\omega)t'}dt'$$

$$= \frac{1}{2}\left[\frac{e^{i(\omega_{21}+\omega)t}-1}{\omega_{21}+\omega} + \frac{e^{i(\omega_{21}-\omega)t}-1}{\omega_{21}-\omega}\right].$$

$$P(E_2,t) = \frac{1}{4\hbar^2}\varepsilon^2|\langle\psi_2|V_0|\psi_1\rangle|^2\left|\left[\frac{e^{i(\omega_{21}+\omega)t}-1}{\omega_{21}+\omega} + \frac{e^{i(\omega_{21}-\omega)t}-1}{\omega_{21}-\omega}\right]\right|^2.$$

Assume $E_2 > E_1$. $P(E_2,t)$ is small, unless the denominator of one of the terms becomes very small. This can only happen for the second term where $\omega_{21} \simeq \omega$; then

$$P(E_2,t) = \frac{1}{4\hbar^2}\varepsilon^2|\langle\psi_2|V_0|\psi_1\rangle|^2 \sin^2\tfrac{1}{2}(\omega_{21}-\omega)t\left(\frac{4}{(\omega_{21}-\omega)^2}\right)$$

$$= \frac{1}{\hbar^2}\varepsilon^2|\langle\psi_2|V_0|\psi_1\rangle|^2\frac{\sin^2\frac{1}{2}(\omega_{21}-\omega)t}{(\omega_{21}-\omega)^2}.$$

problem 4, solution

a) Postulate of QM: $[q,p] = i\hbar$; q, p operators.

$[q,p^2] = [q,p]p + p[q,p] = i\hbar 2p$; commutator algebra: $[A,BC] = B[A,C] + [A,B]C$.

Assume $[q,p^n] = i\hbar n p^{n-1}$ holds for $n = m$.

$[q,p^{m+1}] = [q,pp^m] = p[q,p^m] + [q,p]p^m$

$= p\, i\hbar m p^{m-1} + i\hbar p^m = i\hbar(m+1)p^m$.

Then it also holds for $n = m+1$.

Since it holds for $n=1$ and $n=2$, it holds for all n.

b) $[q,F(p)] = [q, \sum_n f_n p^n] = \sum_n f_n [q,p^n] = \sum_n f_n i\hbar n p^{n-1}$

$= i\hbar \sum_n n f_n p^{n-1} = i\hbar \frac{d}{dp} F(p)$.

c) $[q,p^2 F(q)] = p^2 \underbrace{\cancel{[q,F(q)]}}_{0} + [q,p^2]F(q) = i\hbar 2p F(q)$.

(We can also show: $[p, F(q)] = -i\hbar \frac{d}{dq} F(q)$.

These formulas are used in the proof of Ehrenfest's theorem:

$\frac{d}{dt}\langle\vec{R}\rangle = \frac{1}{m}\langle\vec{p}\rangle$ $\quad \frac{d}{dt}\langle\vec{p}\rangle = -\langle\vec{\nabla} V(\vec{R})\rangle$, $\quad$ for $H = \frac{P^2}{2m} + V(\vec{R})$.

Combining: $m\frac{d^2}{dt^2}\langle\vec{R}\rangle = -\langle\vec{\nabla} V(\vec{R})\rangle$.)

problem 5, solution

a) Let $|n\rangle$ and $|\underline{n}\rangle$ denote the eigenstates of the operator that distinguishes a neutron from an antineutron. In the $\{|n\rangle, |\underline{n}\rangle\}$ orthonormal basis the matrix of H is

$$H = \begin{pmatrix} E_n & \alpha \\ \alpha & E_{\underline{n}} \end{pmatrix}.$$

Let us find the eigenvalues and eigenvectors of H.

We may write

$$H = \frac{1}{2}\begin{pmatrix} E_n + E_{\underline{n}} & 0 \\ 0 & E_n + E_{\underline{n}} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(E_n - E_{\underline{n}}) & \alpha \\ \alpha & -\frac{1}{2}(E_n - E_{\underline{n}}) \end{pmatrix} = \frac{1}{2}(E_n + E_{\underline{n}})\, I + \frac{1}{2}(E_n - E_{\underline{n}}) \begin{pmatrix} 1 & \frac{2\alpha}{E_n - E_{\underline{n}}} \\ \frac{2\alpha}{E_n - E_{\underline{n}}} & -1 \end{pmatrix}.$$

Since every vector is an eigenvector of I, the eigenvectors of $\begin{pmatrix} 1 & \frac{2\alpha}{E_n - E_{\underline{n}}} \\ \frac{2\alpha}{E_n - E_{\underline{n}}} & -1 \end{pmatrix}$ are the eigenvectors of H.

Define: $\tan\theta = \dfrac{2\alpha}{E_n - E_{\underline{n}}}$ and find the eigenvalues and eigenvectors of

$$K = \begin{pmatrix} 1 & \tan\theta \\ \tan\theta & -1 \end{pmatrix}, \qquad \begin{vmatrix} 1-\lambda & \tan\theta \\ \tan\theta & -1-\lambda \end{vmatrix} = 0, \quad \lambda^2 - 1 - \tan^2\theta = 0, \quad \lambda^2 = 1 + \tan^2\theta,$$

$$\lambda = \pm \frac{1}{\cos\theta}.$$

For the eigenvectors we have:

$$\lambda = +\frac{1}{\cos\theta}: \quad \left(1 - \frac{1}{\cos\theta}\right) c_1 + \tan\theta\, c_2 = 0 \qquad c_2 = -c_1\left(\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right) = +c_1 \tan\tfrac{1}{2}\theta.$$

$$|c_1|^2 + |c_2|^2 = 1, \quad \frac{c_2}{c_1} = \frac{+\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}, \quad |\psi_+\rangle = \cos\tfrac{1}{2}\theta\, |n\rangle + \sin\tfrac{1}{2}\theta\, |\underline{n}\rangle$$

$$\lambda = -\frac{1}{\cos\theta}: \quad \left(1 + \frac{1}{\cos\theta}\right) c_1 + \tan\theta\, c_2 = 0 \qquad c_1 = -c_2 \frac{\sin\theta}{1 + \cos\theta} = -c_2 \tan\tfrac{1}{2}\theta.$$

$$|c_1|^2 + |c_2|^2 = 1, \quad \frac{c_1}{c_2} = \frac{-\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}, \quad |\psi_-\rangle = -\sin\tfrac{1}{2}\theta\, |n\rangle + \cos\tfrac{1}{2}\theta\, |\underline{n}\rangle$$

The eigenvectors of H are $|\psi_+\rangle$ with eigenvalue $\frac{1}{2}(E_n+E_{\underline{n}})+\frac{1}{2}(E_n-E_{\underline{n}})\frac{1}{\cos\theta} = E_+$ and $|\psi_-\rangle$ with eigenvalue $\frac{1}{2}(E_n+E_{\underline{n}})-\frac{1}{2}(E_n-E_{\underline{n}})\frac{1}{\cos\theta} = E_-$.

We have: $|n\rangle = \cos\frac{1}{2}\theta\,|\psi_+\rangle - \sin\frac{1}{2}\theta\,|\psi_-\rangle$, $|\underline{n}\rangle = \sin\frac{1}{2}|\psi_+\rangle + \cos\frac{1}{2}|\psi_-\rangle$

$|\psi(0)\rangle = |n\rangle$, $|\psi(t)\rangle = e^{-iE_+t/\hbar}\cos\frac{1}{2}\theta\,|\psi_+\rangle - e^{-iE_-t/\hbar}\sin\frac{1}{2}\theta\,|\psi_-\rangle$

The probability of observing an antineutron at time t is $P_{\underline{n}}(t) = |\langle\underline{n}|\psi(t)\rangle|^2$

$\langle n|\psi(t)\rangle = \cos\frac{1}{2}\theta\sin\frac{1}{2}\theta\left(e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}\right)$.

$P_{\underline{n}}(t) = \frac{1}{4}\sin^2\theta\left(2 - \left(e^{i(E_+-E_-)t/\hbar} + e^{i(E_--E_+)t/\hbar}\right)\right) = \frac{1}{2}\sin^2\theta\left(1-\cos\left((E_+-E_-)t/\hbar\right)\right)$.

period of oscillation: $\frac{2\pi}{T} = \frac{|E_+-E_-|}{\hbar}$, $T = \frac{h}{|E_+-E_-|}$.

Let $\vec{B} = B\hat{z}$, $\vec{\mu}\cdot\vec{B} = \mu_z B$. Then $E_n = m+\mu_z B$ and $E_{\underline{n}} = m-\mu_z B$.
$\frac{1}{2}(E_n+E_{\underline{n}}) = m$, $\frac{1}{2}(E_n-E_{\underline{n}}) = \mu_z B$.
$\tan\theta = \frac{\alpha}{\mu_z B}$. $\frac{1}{\cos^2\theta} = 1+\frac{\alpha^2}{\mu_z^2B^2}$. $\sin^2\theta = \frac{\alpha^2}{\mu_z^2B^2+\alpha^2}$.

$E_+ = m+\sqrt{\mu_z^2B^2+\alpha^2}$, $E_- = m-\sqrt{\mu_z^2B^2+\alpha^2}$ $E_+-E_- = 2\sqrt{\mu_z^2B^2+\alpha^2}$

$P_n(t) = \frac{1}{2}\frac{\alpha^2}{\mu_z^2B^2+\alpha^2}\left(1-\cos\left(2\sqrt{\mu_z^2B^2+\alpha^2}\,t/\hbar\right)\right)$.

b) The larger B, the shorter the period of the oscillations. However, the larger B, the smaller the amplitude of the oscillation.