Homework 1, solutions

problem 1, solution

a) $H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(\rho) = -\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right] + V(\rho).$

$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, $L_z^2 = -\hbar^2\frac{\partial^2}{\partial\phi^2}$, $p_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$, $p_z^2 = -\hbar^2\frac{\partial^2}{\partial z^2}$.

$H = \frac{p_\rho^2}{2\mu} + \frac{L_z^2}{2\mu\rho^2} + \frac{p_z^2}{2\mu} + V(\rho)$, where $p_\rho^2 = -\hbar^2\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)$.

L_z depends only on ϕ, p_z only on z, and p_ρ^2 only on ρ. Therefore $[L_z, p_\rho^2] = [L_z, p_z] = [p_\rho^2, p_z] = 0$ and $[H, L_z] = [H, p_z] = 0$.

We can find a common eigenbasis of H, L_z, and p_z.
$\{e^{im\phi}\}$ = eigenbasis of L_z (m = integer),
$\{e^{ikz}\}$ = eigenbasis of p_z.
Any function $\phi(\rho,\phi,z)$ can be written as a linear combination of functions $e^{im\phi}e^{ikz}$. $\phi(\rho,\phi,z) = \sum_{km} a_{km} e^{im\phi}e^{ikz}$.
(Since k is a continuous index, we should really write it as
$\phi(\rho,\phi,z) = \sum_m e^{im\phi}\int a_m(k)\, e^{ikz}\, dk$.)

A function $f(\rho)e^{im\phi}e^{ikz}$ is an eigenfunction of H if
$H f(\rho)e^{im\phi}e^{ikz} = \left[\frac{p_\rho^2}{2\mu} + \frac{L_z^2}{2\mu\rho^2} + \frac{p_z^2}{2\mu} + V(\rho)\right] f(\rho)e^{im\phi}e^{ikz} = E f(\rho)e^{im\phi}e^{ikz}.$

The plane wave e^{ikz} represents a particle moving with uniform speed along the z-axis. The energy associated with this motion is $\frac{\hbar^2k^2}{2\mu}$.

Define $E' = E - \frac{\hbar^2k^2}{2\mu}$. E' is the energy associated with the motion perpendicular to the z-axis.

$$\left[\frac{p_\rho^2}{2\mu} + \frac{m^2\hbar^2}{2\mu\rho^2} + V(\rho)\right] f(\rho) = E' f(\rho).$$

$f(\rho)$ does not depend on k, only on m. We label the different possible solutions for the same m by the index n. We then have

$$\left[\frac{p_\rho^2}{2\mu} + \frac{m^2\hbar^2}{2\mu\rho^2} + V(\rho)\right] f_{nm}(\rho) = E'_{nm} f_{nm}(\rho).$$

b) $\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)+\frac{m^2\hbar^2}{2\mu\rho^2}+V(\rho)\right]f_{nm}(\rho)=E'_{nm}\,f_{nm}(\rho).$

$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}+\Lambda_{nm}-U(\rho)\right]f_{nm}(\rho)=0.$ $\quad E'_{nm}=\frac{\hbar^2\Lambda_{nm}}{2\mu}$, $\quad U(\rho)=\frac{2m}{\hbar^2}V(\rho)$,

c) $\Sigma_y \Rightarrow \rho\to\rho,\ \phi\to(2\pi-\phi),\ z\to z$

$\Sigma_y\,\psi(\rho,\phi,z)=\psi(\rho,2\pi-\phi,z)$ defines Σ_y.

$\Sigma_y H\,\psi(\rho,\phi,z)=H\,\psi(\rho,2\pi-\phi,z)=H\,\Sigma_y\,\psi(\rho,\phi,z)$ for any $\psi(\rho,\phi,z)$, since

$\frac{\partial}{\partial(2\pi-\phi)}=-\frac{\partial}{\partial\phi},\ \frac{\partial^2}{\partial(2\pi-\phi)^2}=\frac{\partial^2}{\partial\phi^2}.$ $\quad [H,\Sigma_y]=0.$

$\Sigma_y L_z\,\psi(\rho,\phi,z)=-\frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(\rho,2\pi-\phi,z)=-L_z\Sigma_y\,\psi(\rho,\phi,z).$ $\quad \Sigma_y L_z+L_z\Sigma_y=\{\Sigma_y,L_z\}=0$

$L_z\,\phi_{nmk}(\rho,\phi,z)=m\hbar\,\phi_{nm}(\rho,\phi,z).$ $\quad \Sigma_y L_z\,\phi_{nmk}(\rho,\phi,z)=m\hbar\,\Sigma_y\,\phi_{nmk}(\rho,\phi,z)=-L_z\Sigma_y\,\phi_{nmk}(\rho,\phi,z)$

$\Sigma_y\,\phi_{nmk}(\rho,\phi,z)$ is an eigenfunction of L_z with eigenvalue $-m\hbar$.

$\phi_{nmk}(\rho,\phi,z)$ and $\Sigma_y\,\phi_{nmk}(\rho,\phi,z)$ are eigenfunctions of H with the same eigenvalue, but their eigenvalues of L_z are $\pm m\hbar$, respectively.

$H\,\phi_{nmk}=E_{nm}\,\phi_{nmk}$, $\quad H\,\Sigma_y\,\phi_{nmk}=\Sigma_y H\,\phi_{nmk}=E_{nm}\,\Sigma_y\,\phi_{nmk}$, $\Rightarrow$

E_{nm} depends only on $|m|$.

This could have been predicted, since the differential equation depends only on m^2.

problem 2, solution

a) The radial equation for the particle inside the box is

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}\right]R(r)+k^2R(r)=0, \text{ with } k^2=\frac{2mE}{\hbar^2}.$$

b) Let $\rho = kr$. After changing to the variable ρ the radial equation becomes

$$\frac{d^2R}{d\rho^2}+\frac{2}{\rho}\frac{dR}{d\rho}+\left[1-\frac{\ell(\ell+1)}{\rho^2}\right]R=0.$$

This is the spherical Bessel equation of order ℓ. It has two linearly independent solutions. We choose these solutions to be $j_\ell(\rho)$ (the spherical Bessel function) and $n_\ell(\rho)$ (the spherical Neumann function).

c) Acceptable solutions of the radial equation are the spherical Bessel functions $j_\ell(\rho)$. They remain finite at the origin.
We therefore have $R_\ell(r)=C j_\ell(kr)$, subject to the boundary condition $R_\ell(r)=0$ at $r=a$.
We need $j_\ell(ka)=0$.

d) The first 3 functions are

$$j_0(\rho)=\frac{\sin\rho}{\rho}, \quad j_1(\rho)=\frac{\sin\rho}{\rho^2}-\frac{\cos\rho}{\rho}, \quad j_2(\rho)=\left(\frac{3}{\rho^3}-\frac{1}{\rho}\right)\sin\rho-\frac{3}{\rho^2}\cos\rho$$

The eigenvalues are given by the zeroes of the $j_\ell(kr)$.

The first zero of $j_0(\rho)$ occurs at $\rho=3.14$, the first zero of $j_1(\rho)$ occurs at 4.49, and the first zero of $j_2(\rho)$ occurs at $\rho=5.76$.

$$ka=3.14 \Rightarrow E_0=\frac{\hbar^2}{2ma^2}(3.14)^2$$

$$ka=4.49 \Rightarrow E_1=\frac{\hbar^2}{2ma^2}(4.49)^2$$

$$ka=5.76 \Rightarrow E_2=\frac{\hbar^2}{2ma^2}(5.76)^2$$