Homework 4, solutions

problem 1, solution

a). For the 1s ground state we have $\vec{L}=0$, $\vec{J}=\vec{L}+\vec{S}=\vec{S}$, $J=\frac{1}{2}$.

$\vec{F}=\vec{J}+\vec{I}$, $I=1$.

The possible values for F are $F=I+J, I-J$, $F=\frac{3}{2},\frac{1}{2}$.

b). For the 2p excited state we have $L=1$.

The possible values for J are $J=L+S, L-S$, $J=\frac{3}{2},\frac{1}{2}$.

$\vec{F}=\vec{J}+\vec{I}$.

If $J=\frac{3}{2}$, then the possible values for F are

$F=J+I, J+I-1, J-I$, $F=\frac{5}{2},\frac{3}{2},\frac{1}{2}$.

If $J=\frac{1}{2}$ then the possible values for F are $I+J, I-J$, $F=\frac{3}{2},\frac{1}{2}$.

So the possible values for F are: $F=\frac{5}{2},\frac{3}{2},\frac{1}{2}$.

problem 2, solution

a) $E_0^{\mu} = -13.6\,\mathrm{eV}\,\dfrac{\bar m_\mu}{\bar m_e}$, $\bar m_\mu = \dfrac{m_\mu m_p}{m_\mu + m_p}$, $\bar m_e = \dfrac{m_e m_p}{m_e + m_p} \approx m_e$

m_p = proton mass.

b) ground state of ordinary hydrogen, H:

$$\psi_0(\vec r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \qquad a_0 = \frac{\hbar^2}{m_e e^2} = \text{Bohr radius.}$$

ground state of H^μ: $\psi_0^\mu(\vec r) = \dfrac{1}{\sqrt{\pi a^3}} e^{-r/a}$, $a = \dfrac{\hbar^2}{\bar m_\mu e^2}$.

Since $\vec p_e = \vec p_\mu$ we have at $t=0$: $\psi(\vec r) = \psi_0^\mu(\vec r) = \sum_{nlm} a_{nlm} \psi_{nlm}(\vec r)$.

$$\psi(\vec r, t) = \sum_{nlm} a_{nlm} \psi_{nlm}(\vec r) e^{-iE_{nlm}t}, \qquad a_{nlm} = \int d^3r\, \psi^*_{nlm}(\vec r)\, \psi_0^\mu(\vec r).$$

The probability of finding the electron in the ground state of ordinary hydrogen is $|a_{000}|^2 = \left|\int d^3r\, \psi_0^*(r)\, \psi_0^\mu(r)\right|^2$.

$$\int d^3r\, \psi_0^*(r)\, \psi_0^\mu(r) = \frac{4}{\sqrt{a_0^3 a^3}} \int_0^\infty r^2 dr\, e^{-r/a_0} e^{-r/a} =$$

$$\frac{4}{\sqrt{a_0^3 a^3}} \int_0^\infty r^2 dr\, e^{-\left(\frac{1}{a_0}+\frac{1}{a}\right) r} = \frac{4}{\sqrt{a_0^3 a^3}} \left(\frac{a_0 a}{a_0 + a}\right)^3 2.$$

$$|a_{000}|^2 = \left| 8 \frac{(a_0 a)^{3/2}}{(a_0 + a)^3} \right|^2 = \left| 8 \frac{(m_e/\bar m_\mu)^{3/2}}{\left(1 + \frac{m_e}{\bar m_\mu}\right)^3} \right|^2 = \left| \frac{8}{\left(\left(\frac{\bar m_\mu}{m_e}\right)^{1/2} + \left(\frac{m_e}{\bar m_\mu}\right)^{1/2}\right)^3} \right|^2$$

$$= \left| \frac{8}{\left(\sqrt{207} + \frac{1}{\sqrt{207}}\right)^3} \right|^2 = 7 \times 10^{-6}.$$

problem 3, solution

(a) $|\psi\rangle$ is an eigenstate of L^2, L_z, and S_z. (also of S^2)

$L^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle \quad \ell = 2 \qquad L^2|\psi\rangle = 6\hbar^2|\psi\rangle.$

$L_z|\psi\rangle = m_\ell \hbar|\psi\rangle \quad m_\ell = 2 \qquad L_z|\psi\rangle = \hbar|\psi\rangle.$

$S_z|\psi\rangle = m_s\hbar|\psi\rangle \quad m_s = \frac{1}{2} \qquad S_z|\psi\rangle = \frac{1}{2}\hbar|\psi\rangle.$

$\langle S_z\rangle = \frac{\hbar}{2}.$

$S_x = \frac{1}{2}(S_+ + S_-)$, $\langle S_x\rangle = 0.$

$\vec{L}\cdot\vec{S} = L_x S_x + L_y S_y + L_z S_z = \frac{1}{2}(L_+S_- + L_-S_+) + L_z S_z$, $\langle \vec{L}\cdot\vec{S}\rangle = \hbar^2.$

$J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$, $\langle J^2\rangle = 6\hbar^2 + \frac{3}{4}\hbar^2 + \hbar^2 = \frac{35}{4}\hbar^2.$

$J_z = L_z + S_z$, $\langle J_z\rangle = \frac{5}{2}\hbar.$

b) $|\psi\rangle$ is an eigenstate of L^2, J^2 and J_z. (also of S^2)

$L^2|\psi\rangle = 6\hbar^2|\psi\rangle$

$J^2|\psi\rangle = j(j+1)\hbar^2|\psi\rangle \quad j = \frac{3}{2} \qquad J^2|\psi\rangle = \frac{15}{4}\hbar^2|\psi\rangle$

$J_z|\psi\rangle = m_j\hbar|\psi\rangle \quad m_j = -\frac{1}{2} \qquad J_z|\psi\rangle = -\frac{1}{2}\hbar|\psi\rangle$

$\langle J^2\rangle = \frac{15}{4}\hbar^2 \qquad \langle J_z\rangle = -\frac{1}{2}\hbar$

$\vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) \qquad \langle \vec{L}\cdot\vec{S}\rangle = \left(\frac{15}{4} - 6 - \frac{3}{4}\right)\frac{\hbar^2}{2} = -\frac{3}{2}\hbar^2$

$$|2\ \tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle = a\,|2\ \tfrac{1}{2}\ 0\ {-\tfrac{1}{2}}\rangle + b\,|2\ \tfrac{1}{2}\ {-1}\ \tfrac{1}{2}\rangle$$

$\ell\ s\ \ j\ m_j \qquad \ell\ s\ m_\ell\ m_s \qquad m_\ell\ m_s$

$$S_z|\psi\rangle = a\left(-\tfrac{1}{2}\hbar\right)|2\ \tfrac{1}{2}\ 0\ {-\tfrac{1}{2}}\rangle + b\tfrac{1}{2}\hbar\,|2\ \tfrac{1}{2}\ {-1}\ \tfrac{1}{2}\rangle \qquad a = \sqrt{\tfrac{2}{5}} \quad b = -\sqrt{\tfrac{3}{5}}$$

Clebsch-Gordon Coefficients

$$\langle S_z\rangle = \frac{-a^2}{2}\hbar + \frac{b^2}{2}\hbar = \frac{1}{5}\frac{\hbar}{2}$$

$$S_x = \tfrac{1}{2}(S_+ + S_-) \qquad S_+|-\rangle = \hbar|+\rangle \qquad S_-|+\rangle = \hbar|-\rangle$$

$$S_x|\psi\rangle = a\frac{\hbar}{2}|2\tfrac{1}{2}\,0\,\tfrac{1}{2}\rangle + b\frac{\hbar}{2}|2\tfrac{1}{2}\,{-1}\,{-\tfrac{1}{2}}\rangle \qquad \langle S_x\rangle = 0$$

problem 4, solution

The matrices M_i represent the ith component of some angular momentum operator $\vec{J}$ in some basis $\{|\alpha, j\, m\rangle\}$. For each set of values (α, j) there are $2j+1$ different values of m. There can be basis vectors with the same value of j but different values of α.

Since there is one eigenvector with $m=2$ there must be just one set of vectors $\{|\alpha, 2\ m\rangle\}$. There are therefore 5 eigenvectors with the eigenvalue $2(2+1)\hbar^2 = 6\hbar^2$ of J^2 (M^2).
This set of vectors produces the eigenvalue $m=1$ just once.
But it occurs 28 times. There must therefore exist 27 sets of vectors $\{|\alpha, 1, m\rangle\}$. There are therefore $27 \times 3 = 81$ eigenvectors with the eigenvalue $1(1+1)\hbar^2 = 2\hbar^2$ of J^2.
We now have $27+1$ vectors that produce the eigenvalue $m=0$.
However it occurs 70 times. There must therefore be $70-28 = 42$ vectors $|\alpha, 0\ 0\rangle$ with eigenvalue 0 of J^2.
The eigenvalue $\frac{3}{2}\hbar$ occurs 8 times. There are therefore 8 sets of vectors $|\alpha, \frac{3}{2}, m\rangle$ and the eigenvalue $\frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2$ occurs $4 \times 8 = 32$ times. This set of vectors produces $m = \frac{1}{2}$ 8 times. Since it occurs 56 times there must be $56-8 = 48$ vectors $|\alpha, \frac{1}{2}, m\rangle$. They produce the eigenvalue $\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$ $48 \times 2 = 96$ times.
Summary:

value	$6\hbar^2$	$\frac{15}{4}\hbar^2$	$2\hbar^2$	$\frac{3}{4}\hbar^2$	0	
times	5	32	81	96	42	sum = 256

problem 5, solution

$\vec{J}=\vec{L}+\vec{S}$, $\ell=2$, $s=1/2$, so the possible values for j are $\frac{5}{2}$ and $3/2$.
$((\ell+s)\geq j\geq|\ell-s|)$.

$H|jm\rangle=(A+B\vec{L}\cdot\vec{S}+C\vec{L}\cdot\vec{L})|j,m\rangle$, $|j,m\rangle=|\ell,s;jm\rangle$
in more explicit notation,

$\vec{L}\cdot\vec{S}=\frac{1}{2}(J^2-L^2-S^2)$.
$\vec{L}\cdot\vec{S}|\ell,s;jm\rangle=\frac{1}{2}\hbar^2(j(j+1)-\ell(\ell+1)-s(s+1))=\frac{1}{2}\hbar^2(j(j+1)-6-\frac{3}{4})$ for $\ell=2, s=1/2$.
$\vec{L}\cdot\vec{L}=L^2$.
$L^2|\ell,s;jm\rangle=\hbar^2\ell(\ell+1)=6\hbar^2$.

$H|jm\rangle=(A+\frac{1}{2}B\hbar^2(j(j+1)-6\frac{3}{4})+6C\hbar^2)|jm\rangle=E_{jm}|jm\rangle$.

For $j=\frac{5}{2}$ $E_{jm}=A+B\hbar^2+6C\hbar^2$, independent of m.

For $j=3/2$ $E_{jm}=A-\frac{3}{2}B\hbar^2+6C\hbar^2$, independent of m.