Homework 5, solutions

problem 1, solution

$$[V^{(1)} \otimes W^{(1)}]^{(K)}_M = \sum_p \sum_q \langle 1,1; p\, q \,|\, K\, M \rangle V_p^{(1)} W_q^{(1)}.$$

$$\underbrace{\overset{\uparrow}{j_1}\ \overset{\uparrow}{j_2}\ \overset{\uparrow}{m_1}\ \overset{\uparrow}{m_2}\ |\ \overset{\uparrow}{j}\ \overset{\uparrow}{m}}_{C^{j}_{m_1 m_2}}$$

Use the table of CG coefficients.

a) $$[V^{(1)} \otimes W^{(1)}]^{(0)}_0 = \langle 11; 1\,{-1} | 00 \rangle V_1^{(1)} W_{-1}^{(1)} + \langle 11; {-1}\,1 | 00 \rangle V_{-1}^{(1)} W_1^{(1)} + \langle 11; 00 | 00 \rangle V_0^{(1)} W_0^{(1)}$$

$$= \frac{1}{\sqrt{3}} V_1^{(1)} W_{-1}^{(1)} + \frac{1}{\sqrt{3}} V_{-1}^{(1)} W_1^{(1)} - \frac{1}{\sqrt{3}} V_0^{(1)} W_0^{(1)}$$

$$= \frac{1}{\sqrt{3}}\frac{1}{2}\left(-V_x W_x - V_y W_y + i V_x W_y - i V_y W_x\right) + \frac{1}{3}\frac{1}{\sqrt{2}}\left(-V_x W_x - V_y W_y - i V_x W_y + i V_y W_x\right) - \frac{1}{3} V_z W_z = \frac{-1}{\sqrt{3}}\left(V_x W_x + V_y W_y + V_z W_z\right) = -\frac{1}{\sqrt{3}} \vec{V} \cdot \vec{W}.$$

b) $$[V^{(1)} \otimes W^{(1)}]^{(1)}_0 = \langle 11; 1\,{-1} | 10 \rangle V_1^{(1)} W_{-1}^{(1)} + \langle 11; {-1}\,1 | 10 \rangle V_{-1}^{(1)} W_1^{(1)} + \langle 11; 00 | 10 \rangle V_0^{(1)} W_0^{(1)}$$

$$= \frac{1}{\sqrt{2}} V_1^{(1)} W_{-1}^{(1)} - \frac{1}{\sqrt{2}} V_{-1}^{(1)} W_1^{(1)} = \frac{i}{\sqrt{2}}\left(V_x W_y - V_y W_x\right) = \frac{i}{\sqrt{2}} (\vec{V} \times \vec{W})_z.$$

$$[V^{(1)} \otimes W^{(1)}]^{(1)}_1 = \langle 11; 01 | 11 \rangle V_0^{(1)} W_1^{(1)} + \langle 11; 10 | 11 \rangle V_1^{(1)} W_0^{(1)}$$

$$= \frac{-1}{\sqrt{2}} V_0^{(1)} W_1^{(1)} + \frac{1}{\sqrt{2}} V_1^{(1)} W_0^{(1)} = \frac{1}{2}\left(V_z W_x - V_x W_z + i V_z W_y - i V_y W_z\right)$$

$$= \frac{-i}{\sqrt{2}}\left((\vec{V} \times \vec{W})_x + i (\vec{V} \times \vec{W})_y\right).$$

$$[V^{(1)} \otimes W^{(1)}]^{(1)}_{-1} = \langle 11;0-1|1-1\rangle V_0^{(1)} W_{-1}^{(1)} + \langle 11;-10|1-1\rangle V_{-1}^{(1)} W_0^{(1)}$$

$$= \frac{1}{\sqrt{2}} V_0^{(1)} W_{-1}^{(1)} - \frac{1}{\sqrt{2}} V_{-1}^{(1)} V_0^{(1)} = \frac{1}{2}\left(V_z W_x - V_x W_z - i V_z W_y + i V_y W_z\right)$$

$$= \frac{i}{2}\left((\vec{V}\times\vec{W})_x - i(\vec{V}\times\vec{W})_y\right).$$

c) $$[V^{(1)} \otimes W^{(1)}]^{(2)}_{2} = \langle 11;11|22\rangle V_1^{(1)} W_1^{(1)} = \underline{\frac{1}{2} V_+ W_+}.$$

$$[V^{(1)} \otimes W^{(1)}]^{(2)}_{-2} = \langle 11;-1-1|2-2\rangle V_{-1}^{(1)} W_{-1}^{(1)} = \underline{\frac{1}{2} V_- W_-}.$$

$$[V^{(1)} \otimes W^{(1)}]^{(2)}_{1} = \langle 11;10|21\rangle V_1^{(1)} W_0^{(1)} + \langle 11;01|21\rangle V_0^{(1)} W_1^{(1)}$$

$$= \underline{-\frac{1}{2}\left(V_+ W_z + V_z W_+\right)}.$$

$$[V^{(1)} \otimes W^{(1)}]^{(2)}_{-1} = \langle 11;-10|2-1\rangle V_{-1}^{(1)} W_0^{(1)} + \langle 11;0-1|2-1\rangle V_0^{(1)} W_{-1}^{(1)}$$

$$= \underline{\frac{1}{2}\left(V_- W_z + V_z W_-\right)}.$$

$$[V^{(1)} \otimes W^{(1)}]^{(2)}_{0} = \langle 11;00|20\rangle V_0^{(1)} W_0^{(1)} + \langle 11;1-1|20\rangle V_1^{(1)} W_{-1}^{(1)}$$

$$+ \langle 11;-11|20\rangle V_{-1}^{(1)} W_1^{(1)}$$

$$= \sqrt{\frac{2}{3}} V_z W_z + \sqrt{\frac{1}{6}}\left(\frac{-1}{2}\right) V_+ W_- + \sqrt{\frac{1}{6}}\left(\frac{-1}{2}\right) V_- W_+$$

$$= \underline{\sqrt{\frac{2}{3}} V_z W_z - \frac{1}{2}\sqrt{\frac{1}{6}}\left(V_+ W_- + V_- W_+\right)}.$$

$[V^{(1)} \otimes W^{(1)}]^{(0)}$ is a scalar operator (irreducible tensor of rank 0).
$[V^{(1)} \otimes W^{(1)}]^{(1)}$ is a vector operator (irreducible tensor of rank 1).
$[V^{(1)} \otimes W^{(1)}]^{(2)}$ is an irreducible tensor operator of rank 2.

problem 2, solution

a) The spherical components of the vector operator $\vec{R}$ are

$R_0 = z$ and $R_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy)$.

The Wigner-Eckart theorem states that for an irreducible tensor operator

$$\langle \alpha' j' m' | T_q^k | \alpha j m \rangle = \langle j, k ; m q | j, k ; j' m' \rangle \frac{\langle \alpha' j' \| T^k \| \alpha j \rangle}{\sqrt{2j'+1}}.$$

$\vec{R}$ is an irreducible tensor operator of rank 1 and

$$\langle n' \ell' m' | R_q | n \ell m \rangle = \langle \ell, 1 ; m q | \ell, 1 ; \ell' m' \rangle \frac{\langle n' \ell' \| \vec{R} \| n \ell \rangle}{\sqrt{2\ell'+1}}$$

$$= C_{mq}^{\ell'} \frac{\langle n' \ell' \| \vec{R} \| n \ell \rangle}{\sqrt{2\ell'+1}}.$$

The matrix elements of R_q are zero unless $\ell' = \ell+1, \ell, \ell-1$ and

$m' = m+q$, $q = 0, \pm 1$.

b) $$\int \psi'(\vec{r})\, z\, \psi(\vec{r})\, d^3r = A' \int d\Omega\, Y_{\ell' m'}^*\, z\, Y_{\ell m} = A \int d\Omega\, Y_{\ell' m'}^*\, Y_{10}\, Y_{\ell m}.$$

$$\int d\Omega\, Y_{\ell m}^*\, Y_{\ell_1 m_1}\, Y_{\ell_2 m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \langle \ell_1 \ell_2 ; 0 0 | \ell_1 \ell_2 ; \ell 0 \rangle \langle \ell_1 \ell_2 ; m_1 m_2 | \ell_1 \ell_2 ; \ell m \rangle.$$

$A \int d\Omega\, Y_{\ell' m'}^*\, Y_{10}\, Y_{\ell m} = 0$ unless $\ell' = \ell \pm 1$ and $m' = m$.

$\int \psi'(\vec{r}) \left(\frac{1}{\sqrt{2}}(x+iy)\right) \psi(\vec{r})\, d^3r = A \int d\Omega\, Y_{\ell' m'}^*\, Y_{11}\, Y_{\ell m} = 0$ unless $\ell' = \ell \pm 1$

and $m' = m+1$.

$\int \psi'(\vec{r}) \left(\frac{1}{\sqrt{2}}(x-iy)\right) \psi(\vec{r})\, d^3r = A \int d\Omega\, Y_{\ell' m'}^*\, Y_{1-1}\, Y_{\ell m} = 0$ unless $\ell' = \ell \pm 1$

and $m' = m-1$.

From the properties of the spherical harmonics we exclude the possibility $\ell' = \ell$. (3j-symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0$ unless $j_1 + j_2 + j_3 = $ even.)

problem 3, solution

The Hamiltonian of the system is independent of time. The eigenfunctions are of the form $\psi(x,t) = \psi(x)e^{-i\omega t}$.

If the upper path is blocked, the eigenvalue equation is $H_1\psi_1(x) = E\psi_1(x)$.
If the lower path is blocked, the eigenvalue equation is $H_2\psi_2(x) = E\psi_2(x)$.
If both path are open, we may write $\psi(x) = a\psi_1(x) + b\psi_2(x)$.

$$\psi_1(x) = C(k)\, e^{i\int^x k_1(x')dx'} \qquad \psi_2(x) = e^{i\int^x k_2(x')dx'}$$

In the interference region, the phase difference between ψ_1 and ψ_2 is

$$\int_{x_0}^{x_0+\ell}\left(\sqrt{\frac{2m(E+\mu_z B)}{\hbar^2}} - \sqrt{\frac{2mE}{\hbar^2}}\right)dx \approx \left(\sqrt{\frac{2mE}{\hbar^2}}\left(1-\frac{\mu_z B}{2E}\right) - \sqrt{\frac{2mE}{\hbar^2}}\right)\ell .$$

Here we have assumed that $V(x)$ differs from 0 only along a section of path 2 starting at x_0 and ending at $x_0+\ell$.
Here $V(x) = -\vec{\mu}\cdot\vec{B}$, where $\vec{\mu}$ is the magnetic moment of the neutron and $\vec{B}$ is the external field. Let $\vec{B} = B\hat{z}$ then $-\vec{\mu}\cdot\vec{B} = -\mu_z B$.

$$\Delta\phi = -\sqrt{\frac{2mE}{\hbar^2}}\,\frac{\mu_z B}{2E}\,\ell . \quad \text{Let } E = \frac{p^2}{2m},\ p = \frac{h}{\lambda}. \quad \text{Then}$$

$$\Delta\phi = -\frac{p}{\hbar}\frac{\mu_z B\, m\ell}{p^2} = -\frac{\mu_z B\ell m\lambda}{\hbar h}.$$

$$\Delta\phi_1 = -\frac{\mu_z \ell m\lambda}{\hbar h}B_1, \quad \Delta\phi_2 = -\frac{\mu_z\ell m\lambda}{\hbar h}B_2.$$

$$\Delta\phi_1 - \Delta\phi_2 = 2\pi \Rightarrow B_1 - B_2 = \Delta B = -\frac{2\pi\hbar h}{\mu_z \ell m\lambda}.$$

$$\mu_z = \pm g_n\frac{e\hbar}{2mc}. \quad |\Delta B| = -\frac{2\pi\hbar h\, 2mc}{g_n|e|\,\ell\lambda m\hbar} = \frac{4\pi h mc}{g_n|e|m\ell\lambda}.$$

problem 4, solution

a) $\vec{F} = \vec{I} + \vec{J}$ $\quad F^2 = I^2 + J^2 + 2\vec{I}\cdot\vec{J}$ $\quad \vec{I}\cdot\vec{J} = \frac{1}{2}(F^2 - I^2 - J^2)$

$I = J = \frac{1}{2}$ since electron and proton are spin $\frac{1}{2}$ particles.
F can take on the values 1 and 0. $\vec{I}\cdot\vec{J}$ is a scalar operator, its exspectation value is independent of M_F.
$\langle A\, \vec{I}\vec{J}\rangle_{\frac{1}{2}\frac{1}{2}0} = A\frac{1}{2}(0^2 - 2\,\frac{3}{4})\hbar^2 = -\frac{3}{4}\hbar^2 A.$

$\langle A\, \vec{I}\vec{J}\rangle_{\frac{1}{2}\frac{1}{2}1} = A\frac{1}{2}(2 - 2\frac{3}{4})\hbar^2 = \frac{1}{4}\hbar^2 A.$

b) ΔV is a component of a vector operator. The value of its expectation value will depend on M_F.
We can use the projection theorem. In the subspace $E(K, \frac{1}{2}\frac{1}{2}F)$ we have

$$\vec{\Delta V} = \frac{\langle \vec{J}\cdot\vec{\Delta V}\rangle_{\frac{1}{2}\frac{1}{2}F}}{\langle J^2\rangle_{\frac{1}{2}\frac{1}{2}F}}\,\vec{J}$$

We conclude that for $F=1$ $M_F=0$ the expectation value of $\Delta V = (\vec{\Delta V})_z = 0$.

For $F=1, M_F=1$ $\quad |\frac{1}{2}\frac{1}{2}; \underbrace{1\,1}_{F\,M_F}\rangle = |\frac{1}{2}\frac{1}{2}; \underbrace{\frac{1}{2}\frac{1}{2}}_{m_I\,m_J}\rangle$ and therefore

$\langle\Delta V\rangle = B\frac{\hbar}{2}(g_I + g_J)$. Similarly for $F=1, M_F=-1$, $\langle\Delta V\rangle = -B\frac{\hbar}{2}(g_I + g_J)$

For $F=0$ we have $|\frac{1}{2}\frac{1}{2}, 00\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}\rangle - |\frac{1}{2}\frac{1}{2}, -\frac{1}{2}\frac{1}{2}\rangle\right).$
Therefore $\langle\Delta V\rangle = \frac{1}{2}\left(B\frac{\hbar}{2}(g_J - g_I) + \frac{1}{2}\frac{B}{\hbar}(g_J - g_I)\right) = 0.$