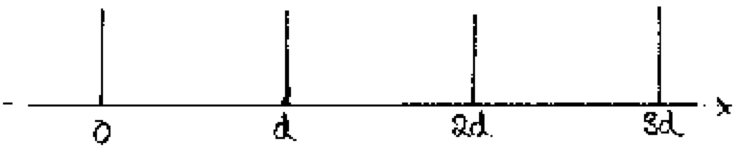
Home work 6, solutions

problem 1, solution



In the region $0<x<d$ we have $\phi(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$, with $\alpha = \sqrt{\frac{2mE}{\hbar^2}}$

Because $V(x+d) = V(x)$ we have $\phi(x) = e^{ikx}\,\upsilon(x)$, with $\upsilon(x) = \upsilon(x+d)$ everywhere.

$$\upsilon(x) = e^{-ikx}\left(Ae^{i\alpha x} + Be^{-i\alpha x}\right) = A\,e^{i(\alpha-k)x} + B\,e^{-i(\alpha+k)x}.$$

At $x=d$ $\phi(x)$ is continuous and $\left.\frac{\partial\phi}{\partial x}\right|_{d+\varepsilon} - \left.\frac{\partial\phi}{\partial x}\right|_{d-\varepsilon} \underset{\varepsilon\to 0}{=} \frac{2mV_0}{\hbar^2}\phi(d)$.

$$\phi(d+\varepsilon) \underset{\varepsilon\to 0}{=} \phi(d-\varepsilon). \quad \phi(d+\varepsilon) \underset{\varepsilon\to 0}{=} e^{ikd}\upsilon(0) = e^{ikd}(A+B), \quad \phi(d-\varepsilon) \underset{\varepsilon\to 0}{=} Ae^{i\alpha d} + Be^{-i\alpha d}.$$

$$A\left(e^{i\alpha d} - e^{ikd}\right) = B\left(e^{ikd} - e^{-i\alpha d}\right).$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{d+\varepsilon} = \left.\frac{\partial}{\partial x}\left(e^{ikx}\upsilon(x)\right)\right|_{d+\varepsilon} = ik\,e^{ikx}\,\upsilon(x)\Big|_{d+\varepsilon} + e^{ikx}\frac{\partial}{\partial x}\upsilon(x)\Big|_{\varepsilon}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{d+\varepsilon} \underset{\varepsilon\to 0}{=} ik\,e^{ikd}(A+B) + e^{ikd}\left(i(\alpha-k)A - i(\alpha+k)B\right).$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{d-\varepsilon} \underset{\varepsilon\to 0}{=} i\alpha A\,e^{i\alpha d} - i\alpha B\,e^{-i\alpha d}.$$

$$\cancel{ik\,e^{ikd}A} + \cancel{ik\,e^{ikd}B} + i\alpha A\,e^{ikd} - \cancel{ikA\,e^{ikd}} - i\alpha B\,e^{ikd} - \cancel{ikB\,e^{ikd}} - i\alpha A\,e^{i\alpha d} + i\alpha B\,e^{-i\alpha d}$$

$$= i\alpha A\left(e^{ikd} - e^{i\alpha d}\right) + i\alpha B\left(e^{-i\alpha d} - e^{ikd}\right) = 2i\alpha A\left(e^{ikd} - e^{i\alpha d}\right)$$

$$= \frac{2mV_0}{\hbar^2}e^{ikd}(A+B) = C\,e^{ikd}\left(A + A\frac{\left(e^{i\alpha d} - e^{ikd}\right)}{\left(e^{ikd} - e^{-i\alpha d}\right)}\right), \quad \text{where } C = \frac{2mV_0}{\hbar^2}.$$

$$2i\alpha e^{ikd}\left(e^{ikd}-e^{-i\alpha d}\right)-2i\alpha e^{i\alpha d}\left(e^{ikd}-e^{-ikd}\right)$$

$$=Ce^{ikd}\left(e^{ikd}-e^{-i\alpha d}+e^{i\alpha d}-e^{ikd}\right)$$

$$2i\alpha\left(e^{ikd}-e^{-i\alpha d}-e^{i\alpha d}+e^{-ikd}\right)=C\left(e^{i\alpha d}-e^{-i\alpha d}\right)$$

$$2i\alpha\left(2\cos kd-2\cos\alpha d\right)=2iC\sin\alpha d$$

$$\frac{C}{2\alpha}\sin\alpha d+\cos\alpha d=\cos kd.$$

Let $\alpha d = x$, then $\frac{Cd}{2x}\sin x+\cos x=\cos kd$.

All energies for which $-1\le\left(\frac{Cd}{2x}\sin x+\cos x\right)\le+1$ are allowed.

problem 2, solution

We assume that the nucleons are confined to a volume $V = \frac{4\pi}{3} R^3 = \frac{4\pi}{3}(1.2)^3 A$.

The density of the protons is $\frac{Z}{V}$ and the density of the neutrons is $\frac{N}{V}$.

We assume that if the number of particles is large enough, the Fermi energy only depends on the density of particles, not on the shape of V.

To keep the calculation simple, we assume the nucleons are confined to a cubical box. We assume that the neutrons and protons are noninteracting particles confined to the cubical box. The potential inside the box is zero, and outside the box it is infinite.
Consider the neutrons in the box.
The energy levels in the box are $E_{lmn} = \frac{(l^2+m^2+n^2)}{2m_n L^2}\pi^2\hbar^2 = \frac{(k_x^2+k_y^2+k_z^2)\hbar^2}{2m_n}$.
Here L is the length of each side of the box.

The number of neutron states in a volume of k-space with $k^2 < k'^2$ is $2\,\frac{1}{8}\,\frac{4\pi}{3}\,k'^3 \Big/ \left(\frac{\pi^3}{L^3}\right)$. $\frac{\pi^3}{L^3}$ is the volume of k-space associated with each state.

Assume N neutrons fill all states with $k^2 < k'^2$. Then $N = \frac{k'^3 L^3}{3\pi^2}$

$k'^3 = 3\frac{N}{L^3}\pi^2 = k_F^3$, $\quad E_F = \frac{\hbar^2 k_F^2}{2m_n} = \left(\frac{3N\pi^2}{L^3}\right)^{2/3}\frac{\hbar^2}{2m_n}$

$\frac{N}{L^3} \rightarrow \frac{N}{V} = \frac{N\,3}{4\pi(1.2)^3 A}\,fm^{-3}$, $\quad E_F = \left(\frac{9N\pi\,(fm^{-3})}{4(1.2)^3 A}\right)^{2/3}\frac{\hbar^2}{2m_n} = \left(4.1\,\frac{N}{A}(fm^{-3})\right)^{2/3}\frac{\hbar^2}{2m_n} = 50\,\frac{N}{A}\,\text{MeV}.$

For the protons in the box we have $E_F = 50\,\frac{Z}{A}$ MeV.

For ^{198}Au, $A = 198$, $Z = 79$, $N = 119$. $\quad E_{F_{neutron}} = 30$ MeV.

$E_{F_{proton}} = 20$ MeV.