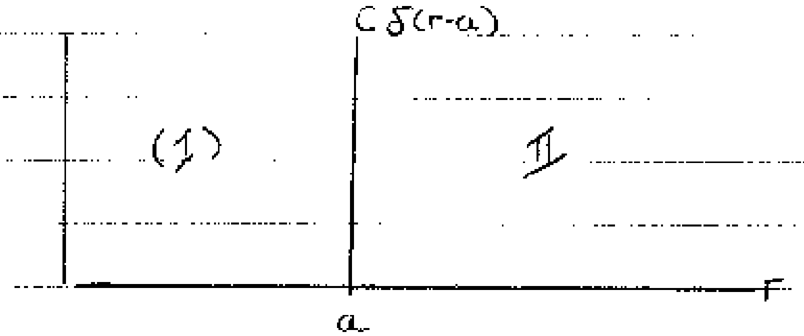
Homework 7, solutions

<u>problem 1, solution</u>

We use the partial wave method. Since $ka \ll 1$, only s-wave scattering is important.

We need to find the phase shift δ_0 for $V(r) = C\,\delta(r-a)$.



Region I: $U(r) = A \sin kr$ $\qquad K^2 = \sqrt{\frac{2mE}{\hbar^2}}$.

Region II: $U(r) = B \sin(kr + \delta_0)$.

Boundary conditions: $U(r)$ is continuous at $r = a$, but $\frac{\partial U}{\partial r}$ is not. $\qquad \frac{\partial^2 U}{\partial r^2} + \frac{2m}{\hbar^2}(E - V(r))U = 0$.

$$\left[\frac{\partial U_2}{\partial r}\bigg|_{a+\varepsilon} - \frac{\partial U_1}{\partial r}\bigg|_{a-\varepsilon}\right]_{\varepsilon\to 0} = \left[-\frac{2m}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon}\left(E - C\delta(r-a)\right)U(r)\,dr\right]_{\varepsilon\to 0} = \frac{2m}{\hbar^2} C\,U(a).$$

We therefore have $\quad A \sin ka = B \sin(ka + \delta_0)$,

$$kA \cos ka = kB\cos(ka + \delta_0) - \frac{2m}{\hbar^2} C A \sin ka.$$

$$k \cot ka + \frac{2m}{\hbar^2} C = k \cot(ka + \delta_0).$$

$$\delta_0 = \cot^{-1}\left(\cot ka + \frac{2m}{\hbar^2 k} C\right) - ka. \qquad \sigma_x = \frac{4\pi}{K^2}\sin^2\delta_0.$$

problem 2, solution

As $E \to 0$, $k \to 0$; $ka \to 0$. If $ka \ll 1$ then the scattering is dominated by s-wave scattering.

Then $\sigma_K(\theta) = \frac{1}{K^2}\sin^2\delta_0(K)$, $\sigma_K = \frac{4\pi}{K^2}\sin^2\delta_0$.

However, when $\delta_0 = \pi$, $\sigma_K(\theta)$ and σ_K are ≈ 0. This is the Ramsauer-Townsend effect. In this case the $\ell=1$ partial wave contribution to the scattering is the dominant contribution and

$\sigma_K(\theta) = \frac{1}{K^2} 9 \sin^2\delta_1 \cos^2\theta$, $\sigma_K = \frac{4\pi}{K^2} 3 \sin^2\delta_1$.

To find what $V_0 a^2$ has to be for $\delta_0 = \pi$, we need to find the expression for δ_0 for a 3-dimensional square well potential.

$V(r) = -V_0 \quad r<a$, $U_{K0}(r) = C_1 \sin\sqrt{K^2+U_0}\, r$, $K^2 = \frac{2mE}{\hbar^2}$, $U_0 = \frac{2mV_0}{\hbar^2}$.

$V(r) = 0 \quad r>a$, $U_{K0}(r) = C_2 \sin(Kr+\delta_0)$.

$U_{K0}(r)$ and $\frac{d}{dr}U_{K0}(r)$ are continuous at $r=a$. This implies

$$\tan(Ka+\delta_0) = \frac{K}{\sqrt{K^2+U_0}}\tan\sqrt{K^2+U_0}\, a.$$

We want $\delta_0 = \pi$. $\tan Ka = \frac{K}{\sqrt{K^2+U_0}}\tan\sqrt{K^2+U_0}\, a$.

As $Ka \to 0$ this becomes $Ka = \frac{K}{\sqrt{U_0}}\tan\sqrt{U_0}\, a$.

$\sqrt{U_0}\, a = \tan\sqrt{U_0}\, a$, $\sqrt{U_0}\, a = n\pi + b$.

$n=0 \quad b=0$.

$n=1 \quad b=1.352$. $\underline{\sqrt{U_0}\, a = 4.494}$ $\underline{\tan\sqrt{U_0}\, a = 4.497}$.

problem 3, solution

a) $\left.\frac{d\sigma^B}{d\Omega}\right|_{\vec{k}_f,\vec{k}_i} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\,V(\vec{r})\right|^2$ with $\vec{q} = \vec{k}_i - \vec{k}_f$ and $|\vec{k}_i| = |\vec{k}_f|$.

$V(\vec{r}) = V_0 e^{-\alpha r}\frac{1}{r}$, $V_0 = -Ze^2$, $\alpha = \frac{1}{a}$.

$$\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\,\frac{e^{-\alpha r}}{r} = \frac{4\pi}{q}\,\frac{q}{\alpha^2+q^2} .$$

$$\left.\frac{d\sigma^B}{d\Omega}\right|_{\vec{k}_f,\vec{k}_i} = \frac{4m^2V_0^2}{\hbar^4}\,\frac{1}{(\alpha^2+q^2)^2} .$$

With $q = 2k\sin\frac{\theta}{2}$ we have $\left.\frac{d\sigma^B}{d\Omega}\right|_{\vec{k}_f,\vec{k}_i} = \frac{4m^2Z^2e^4/\hbar^4}{\left(\frac{1}{a^2}+4k^2\sin^2\frac{\theta}{2}\right)^2}$

b) Let $a \to \infty$, $\alpha \to 0$. Then $\left.\frac{d\sigma^B}{d\Omega}\right|_{\vec{k}_f,\vec{k}_i} = \frac{4m^2Z^2e^4}{\hbar^4}\,\frac{1}{16\sin^4\frac{\theta}{2}\,k^4}$

This is the Rutherford cross section.

c) ratio: $\frac{\left.\frac{d\sigma}{d\Omega}\right|_a}{\left.\frac{d\sigma}{d\Omega}\right|_b} = \left(\frac{q^2}{\left(\frac{1}{a^2}+q^2\right)}\right)^2 = \left(\frac{a^2q^2}{1+a^2q^2}\right)^2 = \left(\frac{x^2}{1+x^2}\right)^2$, $x = aq$.

As $x \to 0$ ratio $\to 0$, $x \to 0 \Rightarrow a \to 0 \Rightarrow \alpha \to \infty$.
The nuclear charge is completely screened. We have a neutral object.

As $x \to \infty$ ratio $\to 1$, $x \to \infty \Rightarrow a \to \infty \Rightarrow \alpha \to 0$
We have a bare nucleus, the screening vanishes.

problem 4, solution

a) $$4\pi\int_0^\infty \rho(r)\, r^2\, dr = \alpha\, 4\pi\int_0^\infty r^2 e^{-\beta r}\, dr = 4\pi\frac{\alpha}{\beta^3}\int_0^\infty r'^2 e^{-r'}\, dr' = \frac{8\pi\alpha}{\beta^3}.$$

$$1.6\times10^{-19}\,C = 4.8\times10^{-10}\,\text{esu} = \frac{8\pi\alpha}{\beta^3}.$$

b) Assume $|\psi(\vec{r})|^2 \propto \rho(r)$, $\quad 4\pi\int_0^\infty |\psi(r)|^2 r^2\, dr = 1.$

$$|\psi(r)|^2 = \frac{\rho(r)\,\beta^3}{8\pi\alpha} = \frac{\beta^3 e^{-\alpha r}}{8\pi}.$$

$$\langle r^2\rangle = \int r^2 |\psi(r)|^2\, d^3r = 4\pi\int_0^\infty r^4\, |\psi(r)|^2\, dr = \frac{1}{2}\beta^3\int_0^\infty r^4 e^{-\beta r}\, dr$$

$$= \frac{1}{2\beta^2}\int_0^\infty r'^4 e^{-r'}\, dr' = \frac{4!}{2\beta^2} = \frac{12}{\beta^2}.$$

c) Assume $\sqrt{\langle r^2\rangle} = 10^{-15}\,m = 10^{-13}\,cm$, $\quad \beta^2 = \frac{12}{\langle r^2\rangle} = 1.2\times10^{27}\,cm^{-2}$,

$$\alpha = 4.8\times10^{-10}\,\text{esu}\,\frac{\beta^3}{8\pi} = 7.9\times10^{29}\,\text{esu cm}^{-3}.$$

problem 5, solution

In the Born approximation we have

$$\sigma_k^B(\theta,\phi) = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r' e^{-i\vec{q}\cdot\vec{r}'}V(\vec{r}')\right|^2, \quad \vec{q} = \vec{k}-\vec{k}', \quad q = 2k\sin\frac{\theta}{2}.$$

$$B\int d^3r' e^{-i\vec{q}\cdot\vec{r}'}\delta(\vec{r}') = B.$$

$$\sigma_k^B(\theta,\phi) = \frac{m^2B^2}{4\pi^2\hbar^4}.$$

The differential scattering cross section is independent of scattering angle and velocity.

$$\sigma_k^B(\theta\phi) = \frac{m^2B^2}{\pi\hbar^4}.$$