Homework 8, solutions

problem 1, solution

In each subspace $\mathcal{E}(E_0 \ell s j)$ we have to diagonalize H_1 to find the energy corrections in first order perturbation theory. $H_1 = \omega_L (L_z + 2S_z)$

From the projection theorem we have

$$L_z = \frac{\langle \vec{L}\cdot\vec{J} \rangle_{E_0 \ell s j}}{j(j+1)\hbar^2} J_z \quad , \quad S_z = \frac{\langle \vec{S}\cdot\vec{J} \rangle_{E_0 \ell s j}}{j(j+1)\hbar^2} J_z \, .$$

$$\vec{L}\cdot\vec{J} = \vec{L}\cdot(\vec{L}+\vec{S}) = \frac{1}{2}(J^2 + L^2 - S^2) \quad , \quad \langle \vec{L}\cdot\vec{J} \rangle_{E_0 \ell s j} = \frac{\hbar^2}{2}\left(j(j+1) + \ell(\ell+1) - s(s+1)\right).$$

$$\vec{S}\cdot\vec{J} = \vec{S}\cdot(\vec{L}+\vec{S}) = \frac{1}{2}(J^2 - L^2 + S^2) \quad , \quad \langle \vec{S}\cdot\vec{J} \rangle_{E_0 \ell s j} = \frac{\hbar^2}{2}\left(j(j+1) - \ell(\ell+1) + s(s+1)\right).$$

$$H_1 = \omega_L \, g_J \, J_z \quad , \quad g_J = \frac{3}{2} + \frac{s(s+1) - \ell(\ell+1)}{2j(j+1)} \, .$$

H_1 is already diagonal with respect to the basis $\{|E_0 \ell s j\rangle\}$ in the subspace $\mathcal{E}(E_0 \ell s j)$. The first order correction to E_0 for the eigenvector $|E_0 \ell s j m\rangle$ is $E_1 = g_J \omega_L m \hbar$.

The magnetic field removes the degeneracy of the multiplet. $(2j+1)$ equidistant levels appear, each one corresponding to one of the possible values of m.

problem 2, solution

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x,y). \qquad V(x,y)=0 \text{ if } 0 \le x \le a \text{ and } 0 \le y \le a.$$
$V(x,y) = \infty$ everywhere else.

$$\phi^0_{np}(x,y) = \frac{2}{a} \sin\frac{n\pi x}{a} \sin\frac{p\pi y}{a}.$$

$$E^0_{np} = \frac{1}{2ma^2}(n^2+p^2)\pi^2\hbar^2,$$

The lowest eigenvalue $E_0^{11} = \frac{\pi^2\hbar^2}{ma^2}$ is non-degenerate. The first excited state is two-fold degenerate.
$$E_0^{21} = E_0^{12} = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}.$$

a) $$E_1^{11} = \langle \phi_{11}|W|\phi_{11}\rangle = \frac{4}{a^2}\int_0^{a/2}\int_0^{a/2} dx\,dy\,\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\,W_0 = \frac{4}{a^2}W_0\frac{(a)^2}{(4)} = \frac{W_0}{4}.$$

$$\left[\int \sin^2 ax\,dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax\right]$$

b) We have to find the matrix elements of W between the degenerate eigenstates.
$$\langle\phi_{12}|W|\phi_{12}\rangle = \frac{4}{a^2}\int_0^{a/2}\int_0^{a/2} dx\,dy\,\sin^2\frac{\pi x}{a}\sin^2\frac{2\pi y}{a}\,W_0 = \frac{4}{a^2}W_0\left(\frac{a}{4}\right)^2 = \frac{W_0}{4}.$$

$$\langle\phi_{21}|W|\phi_{21}\rangle = \frac{W_0}{4}.$$

$$\langle\phi_{12}|W|\phi_{21}\rangle = \frac{4}{a^2}\int_0^{a/2}\int_0^{a/2} dx\,dy\,\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\sin\frac{2\pi y}{a}\,W_0$$
$$= \frac{4}{a^2}W_0\left[\frac{\sin\pi/2}{2\pi/a} - \frac{\sin 3\pi/2}{6\pi/a}\right]^2 = \frac{4}{\pi^2}W_0\left(\frac{1}{2}+\frac{1}{6}\right)^2 = \frac{16}{9\pi^2}W_0,$$

$$\left[\int \sin mx \sin nx\,dx = \frac{\sin(m-n)}{2m-n} - \frac{\sin(m+n)}{2(m+n)}\right]$$

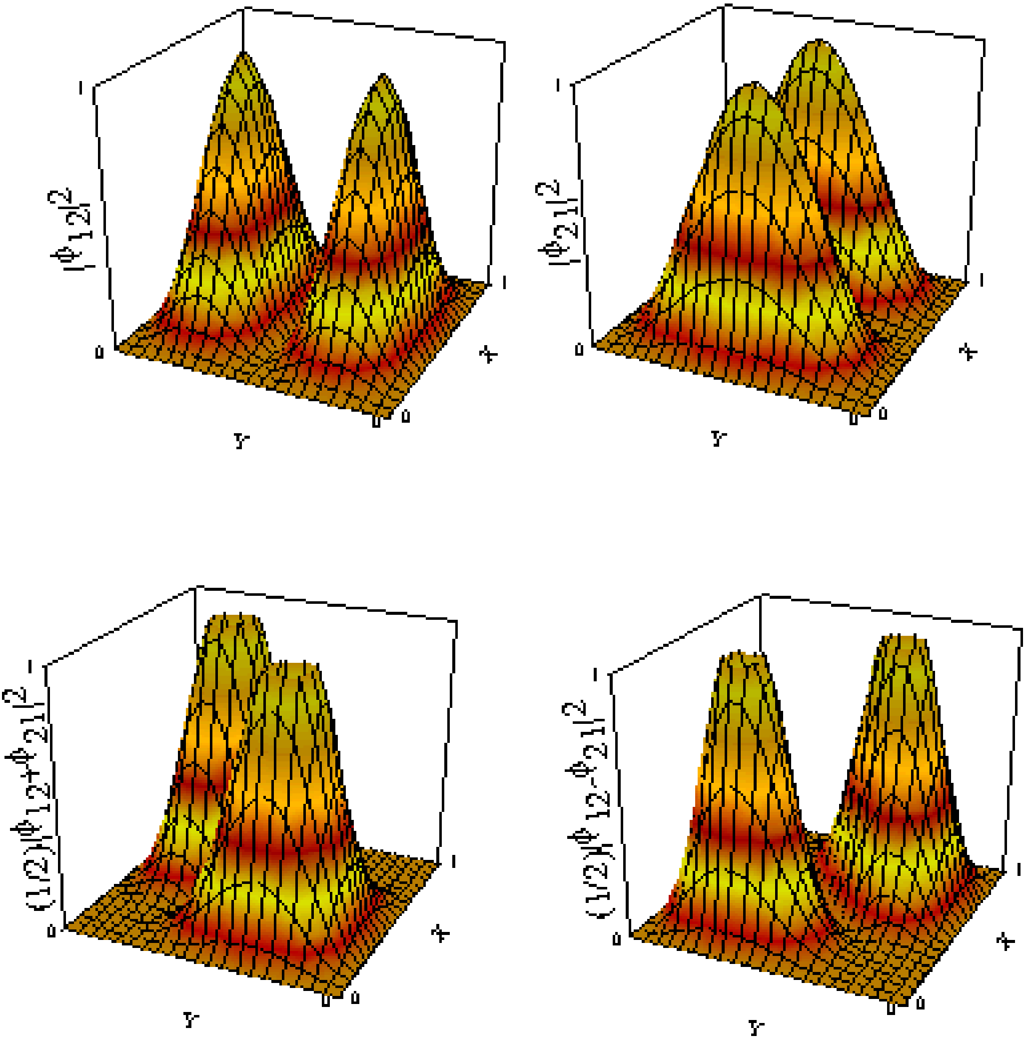
$$\langle\phi_{21}|W|\phi_{12}\rangle = \frac{16}{9\pi^2}W_0.$$

$$\begin{vmatrix} \frac{1}{4} - \frac{E_1}{W_0} & \frac{16}{9\pi^2} \\ \frac{16}{9\pi^2} & \frac{1}{4} - \frac{E_1}{W_0}\end{vmatrix} = 0. \qquad \left(\frac{1}{4} - \frac{E_1}{W_0}\right)^2 - \left(\frac{16}{9\pi^2}\right)^2 = 0.$$
$$\frac{E_1}{W_0} = \frac{1}{4} \pm \frac{16}{9\pi^2}.$$

$$E_1^{(1)} = \left(\frac{1}{4} + \frac{16}{9\pi^2}\right)W_0: \qquad \phi^{(1)} = \frac{1}{\sqrt{2}}\left(\phi_{12} + \phi_{21}\right).$$

$$E_1^{(2)} = \left(\frac{1}{4} - \frac{16}{9\pi^2}\right)W_0: \qquad \phi^{(2)} = \frac{1}{\sqrt{2}}\left(\phi_{12} - \phi_{21}\right).$$

The perturbation removes the degeneracy in first order.


$|\phi_{12}|^2$
$|\phi_{21}|^2$
$(1/2)|\phi_{12}+\phi_{21}|^2$
$(1/2)|\phi_{12}-\phi_{21}|^2$
x
y
0
1

problem 3, solution

a) $\langle \phi_{n'} | X | \phi_n \rangle = A_{n'} A_n \int H_{n'}(\sqrt{\alpha}\, x) H_n(\sqrt{\alpha}\, x)\, e^{-\alpha x^2} dx$

$= A_{n'} A_n \frac{1}{2\sqrt{\alpha}} \int \left[H_{n'+1}(\sqrt{\alpha}x) + 2n' H_{n'-1}(\sqrt{\alpha}x) \right] H_n(\sqrt{\alpha}x)\, e^{-\alpha x^2} dx$

$= \frac{A_{n'}}{A_n} \frac{1}{2\sqrt{\alpha}} \left[\delta_{n'n-1} + 2n' \delta_{n',n+1} \right] = \sqrt{\frac{n}{2\alpha}}\, \delta_{n',n-1} + \sqrt{\frac{n+1}{2\alpha}}\, \delta_{n',n+1}.$

b) $E_1^n = mg \langle \phi_n | X | \phi_n \rangle = 0$ from part a).

c) $E_2^n = m^2 g^2 \sum_{n' \neq n} \frac{|\langle \phi_{n'} | x | \phi_n \rangle|^2}{E_0^n - E_0^{n'}} = m^2 g^2 \left(\frac{n}{2\alpha} \frac{1}{\hbar\omega} - \frac{n+1}{2\alpha} \frac{1}{(-\hbar\omega)} \right)$

$= \frac{-m^2 g^2}{2\alpha \hbar\omega} = -\frac{m^2 g^2}{2m\omega^2} = -\frac{m^2 g^2}{2K},$

d) $x' = x - \frac{mg}{K}$, $\frac{Kx'^2}{2} = \frac{Kx^2}{2} - mgx + \frac{m^2 g^2}{2K}.$

$H = \frac{p^2}{2m} + \frac{Kx'^2}{2} - \frac{m^2 g^2}{2k}$; $E = (n + \tfrac{1}{2})\hbar\omega - \frac{m^2 g^2}{2k}.$