Homework 9 solutions

problem 1, solution

H_0 commutes with L^2, S^2, and J^2 and common eigenstates can be found. The eigenvalues of H_0 can depend on ℓ, but not on s and j.

We denote the common eigenbasis of H_0, L^2, S^2, and J^2 by $\{|n\ell s; j m_j\rangle\}$.

The first order energy correction due to the perturbation is $\langle n\ell s; j m_j | H' | n\ell s; j m_j\rangle = \langle H'\rangle_{n\ell j}$

$\vec{J} = \vec{L} + \vec{S}$ $\quad \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)$

$\langle H'\rangle_{n\ell j} = \frac{1}{2}A\left(j(j+1) - \ell(\ell+1) - s(s+1)\right)$, independent of m_j.

$j = \ell + \frac{1}{2}$ or $j = \ell - \frac{1}{2}$. There are $2j+1$ terms for each j.

The average perturbation is

$(2(\ell+\frac{1}{2})+1)\frac{1}{2}A\left((\ell+\frac{1}{2})(\ell+\frac{3}{2}) - \ell(\ell+1) - \frac{3}{4}\right) +$

$(2(\ell-\frac{1}{2})+1)\frac{1}{2}A\left((\ell-\frac{1}{2})(\ell+\frac{1}{2}) - \ell(\ell+1) - \frac{3}{4}\right) = A\ell^2 + A\ell - A\ell - A\ell^2 = 0$

problem 2, solution

a) The first term in is the classical relativistic correction.

$$E = \sqrt{m^2c^4 + p^2c^2} = mc^2\sqrt{1+\frac{p^2c^2}{m^2c^4}}.$$

$$\sqrt{1+x} = 1+\tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots.$$

$$E = mc^2\left(1+\frac{p^2c^2}{2m^2c^4} - \frac{1}{8}\frac{p^4c^4}{m^4c^8}\right) = mc^2 + \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2}.$$

The second term is the Darwin term. It arises when we make a nonrelativistic approximation to the Dirac equation.

The third term is the spin-orbit interaction for the electron. In the electron's rest frame, the proton orbits the electron. This moving charge produces a magnetic field, and the electron's magnetic moment interacts with this magnetic field.

b) The eigenbasis $\{|n \ell s; j m_j\rangle\}$ is most convenient because in this basis W_f is diagonal.

c) The first term is the spin-orbit interaction for the proton. The proton's magnetic moment interacts with the magnetic field of the orbiting electron.

The 2nd and 3rd terms represent the interaction between two magnetic dipoles. They represent the energy of one magnetic dipole in the field of another. Since the electron's and proton's dipole moments are a consequence of their spin, this is called the spin-spin interaction.

d) No, QED corrections raise the $2S_{1/2}$ with respect to the $2P_{1/2}$ level by a quantity called the Lamb shift.

problem 3, solution

$$H = \frac{p^2}{2\mu} + V(r), \quad r = |\vec{r}_1 - \vec{r}_2|, \quad V(r) = Cr^4, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Use a trial function $\psi(\vec{r}) = R(r)\, Y_{00}(\theta\phi) = A e^{-\alpha r^2} Y_{00}(\theta\phi)$

Then $A^2 \int e^{-2\alpha r^2} r^2 dr = 1$, $\quad A^2 = \frac{4(2\alpha)^{3/2}}{\sqrt{\pi}}$.

$$\langle H \rangle = -\frac{\hbar^2}{2\mu} \int R(r)\, \nabla^2 R(r)\, r^2 dr + C \int R^2(r)\, r^6 dr$$

$$\nabla^2 R(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r R(r)) = \left[4\alpha^2 r^2 - 6\alpha\right] A e^{-\alpha r^2}.$$

$$\langle H \rangle = -\frac{\hbar^2}{2\mu} A^2 \left[-6\alpha \int_0^\infty e^{-2\alpha r^2} r^2 dr + 4\alpha^2 \int_0^\infty r^4 e^{-2\alpha r^2} dr \right] + C A^2 \int_0^\infty r^6 e^{-2\alpha r^2} dr$$

$$= \frac{3}{2} \frac{\hbar^2}{\mu} \alpha + \frac{15C}{16\alpha^2}$$

Set $\frac{\partial \langle H \rangle}{\partial \alpha} = 0$ to adjust α.

$$\frac{\partial \langle H \rangle}{\partial \alpha} = \frac{3}{2} \frac{\hbar^2}{\mu} - \frac{30\, C}{16\, \alpha^3} = 0 \qquad \alpha^3 = \frac{5}{4} \frac{\mu C}{\hbar^2}$$

The estimated ground state energy therefore is

$$\langle H \rangle = \frac{3}{2} \frac{\hbar^2}{\mu} \left(\frac{5}{4} \frac{\mu C}{\hbar^2} \right)^{1/3} + \frac{15C}{16} \left(\frac{5}{4} \frac{\mu C}{\hbar^2} \right)^{-2/3} = 2.42\, \frac{\hbar^{4/3} C^{1/3}}{\mu^{2/3}}.$$

or, using another trial function,

problem 3, solution

$$H = \frac{p^2}{2\mu} + V(r), \quad r = |\vec{r}_1 - \vec{r}_2|, \quad V(r) = Cr^4, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Use as a trial wave function $\psi(\vec{r}) = R(r)\,Y_{00}(\theta\phi) = Ae^{-\alpha r}\,Y_{00}(\theta\phi)$.

Then $A^2\int e^{-2\alpha r} r^2 dr = 1$, $A^2 \frac{2}{(2\alpha)^3} = 1$, $A^2 = 4\alpha^3$.

$$\langle H\rangle = -\frac{\hbar^2}{2\mu}\int R(r)\,\nabla^2 R(r)\,r^2 dr + C\int R^2(r)\,r^6 dr.$$

$$\nabla^2 R(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rR(r)\right) = A\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(re^{-\alpha r}\right) = A\left[\alpha^2 e^{-\alpha r} - \frac{2\alpha}{r}e^{-\alpha r}\right].$$

$$\langle H\rangle = \frac{-A^2\hbar^2}{2\mu}\left[\alpha^2\int e^{-2\alpha r} r^2 dr - 2\alpha\int e^{-2\alpha r} r\,dr\right] + A^2C\int e^{-2\alpha r} r^6 dr$$

$$= \frac{-A^2\hbar^2}{2\mu}\left[\alpha^2\frac{2}{(2\alpha)^3} - 2\alpha\frac{1}{(2\alpha)^2}\right] + A^2C\frac{6!}{(2\alpha)^7}$$

$$= +\frac{A^2\hbar^2}{2\mu}\frac{1}{4\alpha} + A^2C\,\frac{5.625}{\alpha^7} = \frac{\hbar^2}{2\mu}\alpha^2 + C\frac{45}{2\alpha^4}$$

Set $\frac{\partial\langle H\rangle}{\partial\alpha} = 0$ to adjust α.

$$\frac{\partial\langle H\rangle}{\partial\alpha} = \frac{\hbar^2}{\mu}\alpha - C\frac{90}{\alpha^5} = 0 \quad \alpha^6 = C\,90\frac{\mu}{\hbar^2} \qquad \alpha = \left(90C\frac{\mu}{\hbar^2}\right)^{1/6}$$

$$\langle H\rangle = \frac{\hbar^2}{2\mu}\left(90\frac{C\mu}{\hbar^2}\right)^{1/3} + C\frac{45}{2}\left(90C\frac{\mu}{\hbar^2}\right)^{-2/3}$$ is the estimated ground state energy.

$$\langle H\rangle = 3.36\,\frac{\hbar^{4/3}C^{1/3}}{\mu^{2/3}}$$

The "Gaussian" trial function yields a lower value for $\langle H\rangle$.