# Test 1, solutions

## problem 1, solution

a) $\psi(\vec{r}) = \frac{U_{n0}(r)}{r} Y_{00}$ is the form of the eigenfunctions of H

$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + V(r)\right] U_{n0}(r) = E\, U_{n0}(r)$ is the eigenvalue equation for $\ell = 0$.

$0 < r < a$ : $\left[\frac{\partial^2}{\partial r^2} + k^2\right] U_{n0}(r) = 0$, $k^2 = \frac{2mE}{\hbar^2}$.

Boundary conditions; $U_{n0}(r) = 0$ at $r = 0$ and $r = a$.

$U_{n0}(r) = \sin kr$, $ka = n\pi$, $k = \frac{n\pi}{a}$.

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

$\int |\psi_n(\vec{r})|^2 d^3r = 1$ $\quad A^2 \int_0^a \sin^2 \frac{n\pi}{a} r \, dr = 1$ $\quad A = \sqrt{\frac{2}{a}}$.

$\psi_n(\vec{r}) = \sqrt{\frac{2}{a}} \frac{1}{\sqrt{4\pi}} \frac{\sin \frac{n\pi}{a} r}{r}$.

b) $\langle r_n \rangle = \int d^3r \, |\psi_n|^2 r = \frac{2}{a} \int_0^a r \sin^2 \frac{n\pi r}{a} dr = \frac{a}{2}$

$\langle r_n^2 \rangle = \int d^3r \, |\psi_n|^2 r^2 = \frac{2}{a} \int_0^a r^2 \sin^2 \frac{n\pi r}{a} dr = \frac{2a^2}{n^3\pi^3}\left(\frac{n^3\pi^3}{6} - \frac{n\pi}{4}\right)$

$= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$

problem 2, solution

a) The energy of the $n=3$ to $n=2$ transition is $\Delta E = 13.6\,eV\left(\frac{1}{4}-\frac{1}{9}\right) = 1.89\,eV$.
The wavelength of the emitted photon is $\lambda = \frac{hc}{E} = \frac{1.24\times10^4\,\text{Å}\cdot eV}{1.89\,eV} = 6565\,\text{Å}$

b) The energy scales as $\frac{m_e'}{m_e}$ (ignoring reduced-mass effects). $\frac{m_e'}{m_e} = 200$

so $\lambda = 32.8\,\text{Å}$ in this world.

c) In the hypothetical world, the nuclei are still responsible for most of the mass of the atoms. However, atomic dimensions will scale as $\frac{m_e}{m_e'} = \frac{1}{200}$. Densities will be greater by a factor of

$200^3 = 8\times10^6$. We expect densities on the order of $10^7 - 10^8\,g/cm^3$.

d) We expect reaction energies to scale like the atomic energies.
This will lead to reaction temperatures of $\sim 200\times1000\,K = 2\times10^5\,K$.

e) In the hypothetical world $m_p + m_e > m_n$. If the weak interaction is allowed, all protons will capture electrons and become neutrons,
$p + e^- \rightarrow n + \nu$.

problem 3, solutions

a) $j = 3, 2, 1.$

b) $|j_1 j_2; j m\rangle$

$|21;11\rangle = \sqrt{\frac{3}{5}}\,|21;2-1\rangle - \sqrt{\frac{3}{10}}\,|2,1;10\rangle + \sqrt{\frac{1}{10}}\,|2,1;01\rangle.$

$|21;10\rangle = \sqrt{\frac{3}{10}}\,|21;1-1\rangle - \sqrt{\frac{2}{5}}\,|2,1;00\rangle + \sqrt{\frac{3}{10}}\,|2,1;-11\rangle.$

$|21;1-1\rangle = \sqrt{\frac{1}{10}}\,|21;0-1\rangle - \sqrt{\frac{3}{10}}\,|2,1;-10\rangle + \sqrt{\frac{3}{5}}\,|2,1;-21\rangle.$

c) $|j_1 j_2; m_1 m_2\rangle$

$|21;00\rangle = \sqrt{\frac{3}{5}}\,|21;30\rangle - \sqrt{\frac{2}{5}}\,|21;10\rangle.$

d) $\langle J_{1z}\rangle = 2\hbar\frac{3}{5} + \hbar\frac{3}{10} = \frac{3}{2}\hbar.$

$\langle J_{2z}\rangle = -\hbar\frac{3}{5} + \hbar\frac{1}{10} = -\frac{1}{2}\hbar.$

problem 4, solution

a) $E_{vJ} = (v+\frac{1}{2})\hbar\omega - V_0 + Bh\ell(\ell+1)$ with $B = \frac{\hbar}{4\pi\mu a^2}$

$\mu = \frac{M_1 M_2}{M_1+M_2} = \frac{35}{36}$ amu.

For the vibrational-rotational spectrum we have $\Delta v = \pm 1$, $\Delta\ell = \pm 1$.

The frequency of a photon in a band is therefore given by

$\frac{\omega}{2\pi} + 2B(\ell+1) = \nu$, $\ell = 0, 1, 2, \ldots$

Two adjacent lines differ by $\Delta\nu = 2B = \frac{\hbar}{2\pi\mu a^2}$.

Given: $\Delta\frac{\nu}{c} = \frac{\hbar}{c\,2\pi\mu a^2} = 20.9\ \mathrm{cm^{-1}}$

$\Rightarrow \mu a^2 = I = 2.65 \cdot 10^{-47}\ \mathrm{kg\,m^2}$

$a = \sqrt{\frac{I}{\mu}} = 1.29 \times 10^{-10}\ \mathrm{m}$.

b) The equilibrium distances for HCl and DCl are the same. The form of the potential curves are determined by the electronic states.
We therefore have

$(\Delta\frac{\nu}{c})_{DCl} / (\Delta\frac{\nu}{c})_{HCl} = \frac{\mu_{HCl}}{\mu_{DCl}} = \frac{35}{36}\frac{37}{70} = 0.5139.$

$(\Delta\frac{\nu}{c})_{DCl} = 10.7\ \mathrm{cm^{-1}}$.