Addition of 3 angular momenta

Let $\vec{J}_1, \vec{J}_2$, and $\vec{J}_3$ be three angular momentum operators, and let $\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$.
Common eigenvectors of J^2 and J_z are characterized by the quantum numbers j and m.
Let the quantum numbers j_1, j_2, and j_3 be fixed (for example, consider 3 spin ½ particles) and consider the subspace $E(j_1 j_2 j_3\, j m)$ of all states characterized by fixed values of $j_1 j_2 j_3, j$, and m.

J_1^2, J_2^2, J_3^2, J^2 and J_z do not form a C.S.C.O for $E(j_1 j_2 j_3\, j m)$.

We need an additional observable to form a C.S.C.O..

① $\vec{J}_1 + \vec{J}_2 = \vec{J}_{12}$, $\vec{J}_{12} + \vec{J}_3 = \vec{J}$ ② $\vec{J}_2 + \vec{J}_3 = \vec{J}_{23}$, $\vec{J}_{23} + \vec{J}_1 = \vec{J}$ ③ $\vec{J}_1 + \vec{J}_3 = \vec{J}_{13}$, $\vec{J}_{13} + \vec{J}_2 = \vec{J}$

Choosing this additional observable means choosing an angular momentum coupling scheme.
The common eigenvectors of J_1^2, J_2^2, J_3^2, J^2 and J_z and this additional observable form a basis for $E(j_1 j_2 j_3\, j m)$.
Let this additional observable be J_{12}^2 ($\vec{J}_{12} = \vec{J}_1 + \vec{J}_2$). Then the basis vectors are $\{|j_1 j_2 [j_{12}]; j_3\rangle\}$. (We do not have to write $|j_1 j_2 [j_{12}]; j_3; j m\rangle$ because all vectors in $E(j_1 j_2 j_3\, j m)$ have quantum number j and m.)
Similarly, if the additional observable is J_{13}^2 then the basis vectors are $\{|j_1 j_3 [j_{13}]; j_2\rangle\}$, and if the additional observable is J_{23}^2 then the basis vectors are $\{|j_1; j_2 j_3 [j_{23}]\rangle\}$.

We can transform from one coupling scheme to another coupling scheme.

For example

$$|j_1; j_2 j_3 [j_{23}]\rangle = \sum_{j_{12}} |j_1 j_2 [j_{12}]; j_3\rangle \underbrace{\langle j_1 j_2 [j_{12}]; j_3 | j_1; j_2 j_3 [j_{23}]\rangle}_{[j_{12}\, j_{23}]^{1/2}\, W(j_1 j_2 J j_3;\, j_{12} j_{23})}$$

defines the Racah W-coefficient. Here $[j_1, j_2, \ldots] = (2j_1+1)(2j_2+1)\cdots$.

Similarly,

$$|j_1 j_3 [j_{13}]; j_2\rangle = \sum_{j_{12}} |j_1 j_2 [j_{12}]; j_3\rangle \underbrace{\langle j_1 j_2 [j_{12}]; j_3 | j_1 j_3 [j_{13}]; j_2\rangle}_{[j_{12}\, j_{23}]^{1/2}\, W(j_{12}\, j_3\, j_2\, j_{13};\, j\, j_1)}$$

<u>Definition of the Racah W-coefficients:</u>

$$\langle j_1 j_2 [j'] j_3 ; jm | j_1 ; j_2 j_3 [j''] ; jm\rangle = \sqrt{(2j'+1)(2j''+1)}\; W(j_1 j_2 j j_3 ; j' j'')$$

$$\langle j_1 j_2 [j'] j_3 ; jm | j_1 j_3 [j''] j_2 ; jm\rangle = \sqrt{(2j'+1)(2j''+1)}\; W(j' j_3 j_2 j'' ; j j_1)$$

We can express the Racah W-coefficients in terms of the Clebsch-Gordon coefficients.

We may write

$$|j_1 ; j_2 j_3 [j_{23}]\rangle = \sum_{m_1 m_2 m_3} |j_1 m_1 , j_2 m_2 , j_3 m_3\rangle \langle j_1 m_1 , j_2 m_2 j_3 m_3 | j_1 ; j_2 j_3 [j_{23}]\rangle$$

$$= \sum_{\substack{m_1 m_2 m_3 \\ m_{23}}} |j_1 m_1 , j_2 m_2 , j_3 m_3\rangle \langle j_1 m_1 j_2 m_2 j_3 m_3 | j_1 m_1 j_{23} m_{23}\rangle \langle j_1 m_1 j_{23} m_{23} | j_1 ; j_2 j_3 [j_{23}]\rangle .$$

But we also have

$$|j_1 ; j_2 j_3 [j_{23}]\rangle = \sum_{j_{12}} \underbrace{|j_1 j_2 [j_{12}] ; j_3\rangle}\; [j_{12}\, j_{23}]^{1/2}\, W(j_1 j_2 j j_3 ; j_{12} j_{23}) .$$

$$\sum_{\substack{m_1 m_2 m_3 \\ m_{12}}} |j_1 m_1 j_2 m_2 j_3 m_3\rangle \langle j_1 m_1 j_2 m_2 j_3 m_3 | j_{12} m_{12} j_3 m_3\rangle \langle j_{12} m_{12} j_3 m_3 | j_1 j_2 [j_{12}] , j_3\rangle$$

The vectors $|j_1 m_1 j_2 m_2 j_3 m_3\rangle$ with different m_1, m_2, or m_3 are orthogonal. We can therefore equate the coefficients with equal m_1, m_2 and m_3.

$$\sum_{m_{23}} \underbrace{\langle j_1 m_1 j_2 m_2 j_3 m_3 | j_1 m_1 j_{23} m_{23}\rangle}_{\langle j_2 m_2 j_3 m_3 | j_{23} m_{23}\rangle = C^{j_{23}}_{m_2 m_3}} \underbrace{\langle j_1 m_1 j_{23} m_{23} | j_1 ; j_2 j_3 [j_{23}]\rangle}_{\langle j_1 m_1 j_{23} m_{23} | jm\rangle = C^{j}_{m_1 m_{23}}}$$

$$= \sum_{j_{12}} [j_{12}\, j_{23}]^{1/2}\, W(j_1 j_2 j j_3 ; j_{12} j_{23}) \sum_{m_{12}} \underbrace{\langle j_1 m_1 j_2 m_2 j_3 m_3 | j_{12} m_{12} j_3 m_3\rangle}_{\langle j_1 m_1 j_2 m_2 | j_{12} m_{12}\rangle = C^{j_{12}}_{m_1 m_2}} \underbrace{\langle j_{12} m_{12} j_3 m_3 | j_1 j_2 [j_{12}] ; j_3\rangle}_{\langle j_{12} m_{12} j_3 m_3 | jm\rangle = C^{j}_{m_{12} m_3}}$$

Now we can use the orthogonality of the CG coefficients:

$$\sum_{m_1m_2} \langle j_1j_2;j'm' | j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2 | j_1j_2;jm\rangle = \sum_{m_1m_2} C^{j}_{m_1m_2} C^{j'}_{m_1m_2} = \delta_{jj'}\delta_{mm'},$$

and

$$\sum_{jm} \langle j_1j_2;m_1'm_2' | j_1j_2;jm\rangle\langle j_1j_2;jm | j_1j_2;m_1m_2\rangle = \sum_{jm} C^{j}_{m_1'm_2'} C^{j}_{m_1m_2} = \delta_{m_1'm_1}\delta_{m_2'm_2}$$

(The CG coefficients are real.)

Therefore

$$\sum_{\substack{m_1m_2m_3\\ m_{23}m_{12}}} C^{j_{23}}_{m_2m_3} C^{j}_{m_1m_{23}} C^{j_{12}'}_{m_1m_2} C^{j}_{m_{12}m_3}$$

$$= \sum_{\substack{m_1m_2m_3\\ j_{12}m_{12}}} [j_{12},j_{23}]^{1/2} W(j_1j_2jj_3;j_{12}j_{23}) \underbrace{C^{j_{12}}_{m_1m_2} C^{j_{12}'}_{m_1m_2}}_{\text{The sum yields } \delta_{j_{12}j_{12}'}.} \underbrace{C^{j}_{m_{12}m_3} C^{j}_{m_{12}m_3}}_{\text{The sum yields 1.}}$$

$$= [j_{12}' \, j_{23}]^{1/2} W(j_1j_2j\,j_3;j_{12}'j_{23}).$$

The Racah coefficient is independent of the values of m_1, m_2, m_3, m_{12}, and m_{23} and is a function of its six arguments only.

The W-coefficients can be expressed in terms of the <u>6-j symbols</u>.

$$W(j_1j_2l_1l_2;j_3l_3) = (-1)^{-j_1-j_2-l_1-l_2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

6-j symbol calculators can be found on the WWW.

Example:

For the transition from the scheme
$\vec{l}_1+\vec{l}_2=\vec{L}'$, $\vec{s}_1+\vec{s}_2=\vec{S}'$, $\vec{L}'+\vec{l}_3=\vec{L}$, $\vec{S}'+\vec{s}_3=\vec{S}$, to the scheme
$\vec{l}_2+\vec{l}_3=\vec{L}''$, $\vec{s}_2+\vec{s}_3=\vec{S}''$, $\vec{l}_1+\vec{L}''=\vec{L}$, $\vec{s}_1+\vec{S}''=\vec{S}$ we have

$$\langle l_1s_1\, l_2s_2\,[L'S'];l_3s_3:LS | l_1s_1;l_2s_2\,l_3s_3\,[L''S'']:LS\rangle =$$
$$[L'L''S'S'']^{1/2}\, W(l_1l_2Ll_3;L'L'')\, W(s_1s_2Ss_3;S'S'').$$

Addition of 4 angular momenta

For the full description of a state, it is necessary to give the quantum numbers j and m, as well as the angular momenta of two-particle or three-particle subsystems.

Consider the subspace $E(j_1 j_2 j_3 j_4 j m)$.

A state in this subspace may be described by the quantum numbers $j_1 j_2 [j'] j_3 [j''] j_4 ; j m$ or $j_1 j_2 j_3 [j'] [j''] j_4 ; j m$ or $j_1 j_2 [j'] ; j_3 j_4 [j''] ; j m$ or $j_1 j_3 [j'] ; j_2 j_4 [j''] ; j m$, and so on.

Consider, for example, the transformation from the coupling scheme $j_1 j_2 [j_{12}] ; j_3 j_4 [j_{34}] ; j m$ to the scheme $j_1 j_3 [j_{13}] ; j_2 j_4 [j_{24}] ; j m$.

$$|j_{12}\, j_{34} ; j m\rangle = \sum_{j_{13} j_{24}} |j_{13}\, j_{24} ; j m\rangle \underbrace{\langle j_{13}\, j_{24} ; j m | j_{12}\, j_{34} ; j m\rangle}_{\text{recoupling coefficient } \Delta} .$$

The recoupling coefficient Δ can be written in terms of the Racah W-coefficients or the 6-j symbols.

$$\Delta = [j_{12}\, j_{34}\, j_{13}\, j_{24}]^{1/2} \sum_{j'} (-1)^{2j'} (2j'+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_{34} & j & j' \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_{34} \\ j_2 & j' & j_{24} \end{Bmatrix} \begin{Bmatrix} j_{13} & j_{24} & j \\ j' & j_1 & j_3 \end{Bmatrix} .$$

We define the 9-j symbol as

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \sum_{j'} (-1)^{2j'} (2j'+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_{34} & j & j' \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_{34} \\ j_2 & j' & j_{24} \end{Bmatrix} \begin{Bmatrix} j_{13} & j_{24} & j \\ j' & j_1 & j_3 \end{Bmatrix} .$$

Then $\Delta = [j_{12}\, j_{34}\, j_{13}\, j_{24}]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix}$.

9-j symbol calculators can be found on the WWW.