Test 1, solutions

Problem 1, solution

a) \( \Psi(r) = \frac{U_n(r^3)}{r} \) in the form of the eigenfunctions of \( H \)

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) \right] U_n(r) \cdot E_n U_n(r) \text{ in the eigenvalue equation for } E = 0,
\]

\( 0 < r < a \) : \( \frac{d^2}{dr^2} + k^2 U_n(r) = 0 \), \( k^2 = \frac{2mE}{\hbar^2} \).

Boundary conditions: \( U_n(r) = 0 \) at \( r = 0 \) and \( r = a \).

\( U_n(r) = \sin kr, \quad ka = \pi n, \quad k = \frac{n\pi}{a} \).

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2m a^2} \]

\[
\int |\Psi_n(r)|^2 d^3r = 1 \quad A \int_0^a \sin^2 \frac{knr}{a} r dr = 1 \quad A = \frac{\sqrt{2}}{a}.
\]

\( \Psi_n(r) = \frac{\sqrt{2}}{a} \frac{1}{\sqrt{n!}} \frac{\sin \frac{knr}{a}}{r} \).

b) \( \langle r \rangle = \int d^3r |\Psi|^2 r = \frac{2}{a} \int_0^a r \sin^2 \frac{knr}{a} dr = \frac{a}{2} \)

\( \langle r^2 \rangle = \int d^3r |\Psi|^2 r^2 = \frac{2}{a^2} \int_0^a r^2 \sin^2 \frac{knr}{a} dr = \frac{2a^3}{n^3\pi^3} \left( \frac{n^3\pi^2}{6} - \frac{n^3}{4} \right) \)

\[ = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} \]
a) The energy of the $n = 3 \to n = 2$ transition is $\Delta E = 13.6 \text{eV} \cdot \left( \frac{3}{4} - \frac{1}{2} \right) = 1.89 \text{eV}$.

The wavelength of the emitted photon is $\lambda = \frac{hc}{\Delta E} \approx \frac{1.24 \times 10^{-5} \text{eV} \cdot \text{cm}}{1.89 \text{eV}} \approx 656.5 \text{Å}$.

b) The energy scales as $\frac{m_e'}{m_e}$ (ignoring reduced-mass effects), $\frac{m_e'}{m_e} = 200$.

So $\lambda \approx 32.8 \text{Å}$ in this world.

c) In the hypothetical world, the nuclei are still responsible for most of the mass of the atoms. However, atomic dimensions will scale as $\frac{m_e}{m_e'} = \frac{1}{200}$. Density will be greater by a factor of $200^3 \approx 8 \times 10^6$. We expect densities on the order of $10^3 - 10^8 \text{g/cm}^3$.

d) We expect reaction energies to scale like the atomic energies. This will lead to reaction temperatures of $\sim 200 \times 10^6 \text{K} \approx 2 \times 10^8 \text{K}$.

c) In the hypothetical world $m_p + m_e > m_\alpha$. If the weak interaction is allowed, all protons will capture electrons and become neutrons:

$p + e^- \to n + \nu$. 
a) \( j = 3, 2, 1 \).

b) \[ |121; 11 \rangle = \sqrt{\frac{3}{5}} |121; 2; 1 \rangle - \sqrt{\frac{3}{10}} |121; 1; 0 \rangle + \frac{1}{10} |121; 0; 1 \rangle. \]

\[ |121; 10 \rangle = \sqrt{\frac{3}{10}} |121; 1; 1 \rangle - \frac{2}{5} |121; 0; 0 \rangle + \sqrt{\frac{3}{10}} |121; 0; -1 \rangle. \]

\[ |121; 1 - 1 \rangle = \frac{1}{10} |121; 0; -1 \rangle - \sqrt{\frac{3}{10}} |121; 1; -1 \rangle + \sqrt{\frac{3}{5}} |121; 1; -2 \rangle. \]

c) \[ |121; m; m_2 \rangle \]

\[ |121; 0 0 \rangle = \frac{1}{2} |121; 30 \rangle - \frac{2}{5} |121; 10 \rangle. \]

d) \[ \langle J_{z} \rangle = 2 \hbar \frac{3}{5} + \hbar \frac{2}{10} = \frac{3}{2} \hbar, \]

\[ \langle J_{z} \rangle = -\hbar \frac{3}{5} + \hbar \frac{1}{10} = -\frac{1}{2} \hbar. \]
problem 4, solution

a) \( E_{v_2} = (v+\frac{1}{2})h \omega - V_0 + Bte^{(e+1)} \) with \( B = \frac{k}{4\pi \hbar a^2} \)

\[
u = \frac{M_H H}{M + M_H} = \frac{35}{36} \text{amu.}
\]

For the vibrational-rotational spectrum we have
\( \Delta v = 1, \Delta e = -1. \)

The frequency of a photon in a band is therefore given by

\[
\frac{\omega}{2\pi} + 2B(e+1) = \nu, \quad e = 0, 1, 2, \ldots
\]

Two adjacent lines differ by \( \Delta \nu = 2B = \frac{k}{2\pi \hbar a^2} \).

Given: \( \Delta \frac{\nu}{e} = \frac{k}{c \hbar a^2} = 2.09 \text{ cm}^{-1} \)

\[
\Rightarrow \quad a^2 = f = 2.05 \times 10^{-48} \text{ kg m}^2
\]

\[
a = \sqrt{\frac{f}{\mu}} = 1.29 \times 10^{-10} \text{ m.}
\]

b) The equilibrium distances for HCl and DCl are the same. The form of the potential curves are determined by the electronic states.

We therefore have

\[
\frac{(\Delta \frac{\nu}{e})_{DCl}}{(\Delta \frac{\nu}{e})_{HCl}} = \frac{M_{HCl}}{M_{DCl}} = \frac{35}{36} \cdot \frac{37}{70} = 0.5139.
\]

\[
(\Delta \frac{\nu}{e})_{DCl} = 10.7 \text{ cm}^{-1}.
\]